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Author: Paulina Wiśniewska

Asymptotic invariants of configurations of points determined by complex reflection groups

Supervisor: prof. dr hab. Tomasz Szemberg

Associate supervisor: dr hab. Justyna Szpond

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Autor: Paulina Wiśniewska

Asymptotyczne niezmienniki konfiguracji punktów zadane przez zespolone grupy odbić

Promotor: prof. dr hab. Tomasz Szemberg

Promotor pomocniczy: dr hab. Justyna Szpond

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Introduction

For an arbitrary non-trivial homogeneous ideal I one defines its initial degree $\alpha(I)$ as the minimal number d such that the homogeneous part of I of degree d is non-zero. Waldschmidt, working on some problems in complex analysis, introduced the asymptotic version of the initial degree. However the term "Waldschmidt constant" was coined around 2010 by Dumnicki and Harbourne. The symbol $\widehat{\alpha}$, as paralleling the notation for asymptotic cohomology functions, was proposed by Szemberg. The formal definition of the Waldschmidt constant of a homogeneous ideal in a ring of polynomials is

$$\widehat{\alpha}(I) = \inf \frac{\alpha(I^{(m)})}{m},$$

where $I^{(m)}$ is the m-th symbolic power of I. As already mentioned, this invariant was studied much earlier in complex analysis in connection with higher dimensional variants of the Schwarz Lemma. Waldschmidt constants are closely related to Seshardi constants and thus they are expectedly difficult to compute. Around 1980 Chudnovsky stated a conjecture predicting interesting lower bounds on the growth order of the sequence determining the Waldschmidt constant and therefore the constant itself. Research presented here was partly motivated by an attempt to test this conjecture for highly symmetric configurations of points.

Thus the first purpose of this thesis is to compute the values of Waldschmidt constants for some configurations of points determined by complex reflection groups. It is expected that these values are rational and that they satisfy the Chudnovsky conjecture. The computations were expected to be easier than in the general case because of the symmetries characterizing reflection arrangements. Partial results in this direction have been obtained recently in [2]. However, that article also shows that one needs new arguments and ideas in this situation. For example there is only an estimate on the Waldschmidt constant of the configuration of points in $\mathbb{P}^2$ determined by the Klein arrangement (G_{30} in the Sheppard-Todd classification [27]) provided in [2]. Our work shows that additional issues arise when passing from configurations in the plane to higher dimensional projective spaces.

Partial results on values of Waldschmidt constants of symmetric sets of points are scattered in the literature. They concern mainly sets of points in the projective plane. The purpose of this project was to extend the collection of available examples by some configurations in higher dimensional projective spaces. We apply methods from commutative algebra and algebraic geometry. Some of them have been developed in Krakow by Dumnicki, Malara, Szemberg, Szpond and Tutaj-Gasińska, among others.

In our thesis, comprising five chapters, the first chapter introduces essential definitions, examples, and theorems from algebraic geometry and commutative algebra, providing a foundational framework for subsequent discussions.

In chapter two we recall some facts about arrangements associated to finite reflection groups. In this part we consider in particular properties of the D_4 configuration from the point of view of its Dynkin diagram.

The third section of this thesis explores fundamental properties of asymptotic invariants, specifically, Waldschmidt constants and resurgences, offering a comprehensive analysis of available tool allowing their study.

The Waldschmidt constants of symmetric sets of points in projective spaces are the core of chapter four. In particular, we compute the values of Waldschmidt constants for configurations of points determined by some complex reflection groups. We focus on H_3, D_4, B_4, F_4 and H_4 root systems.

In the fifth chapter, we shift our focus to the properties of general projections of symmetric sets of points in projective spaces. We specifically investigate the geproci property, which denotes sets projecting to complete intersection sets of points in the projective plane. Our study centers on the H_4 configuration of points from this perspective, revealing its geproci nature despite not fitting the grid or half-grid categories. It's worth noting that there are currently only three known examples of such geproci sets.

In the Appendix we present some Singular – symbolic algebra program [11] code used to estimate the Waldschmidt constant of the aforementioned configurations. We hope it might be of more general interest, as it can be easily adapted to other configurations of points.

Some additional short Singular scripts are included directly in the appropriate spots of the main text.

Wstęp

Dla dowolnego nietrywialnego jednorodnego ideału $\mathcal{I}$ definiuje się jego stopień początkowy $\alpha(\mathcal{I})$ jako minimalną liczbę d taką, że jednorodna część $\mathcal{I}$ stopnia d jest niezerowa. Waldschmidt, pracując nad niektórymi problemami analizy zespolonej, wprowadził asymptotyczną wersję stopnia początkowego.

Sam termin „stała Waldschmidta” został wprowadzony w 2010 roku przez Dumnickiego i Harbourna. Samo oznaczenie $\widehat{\alpha}$ zostało zaproponowane przez Szemberga. Formalna definicja stałej Waldschmidta wygląda następująco:

Niech dany będzie ideał jednorodny $\mathcal{I}$. Stałą Waldschmidta nazywamy liczbę rzeczywistą

$$\widehat{\alpha}(\mathcal{I}) = \inf \frac{\alpha(\mathcal{I}^{(m)})}{m},$$

gdzie $\mathcal{I}^{(m)}$ oznacza m-tą potęgę symboliczną $\mathcal{I}$.

Jak już wspomniano, niezmiennik ten był badany znacznie wcześniej w analizie zespolonej. Stałe Waldschmidta są blisko spokrewnione ze stałymi Seshardiego i dlatego należy się spodziewać, że ich obliczenie jest bardzo trudne. Około 1980 roku Chudnovsky postawił hipotezę dotyczącą stałej Waldschmidta. Zaprezentowane w niniejszej pracy doktorskiej badania były częściowo motywowane próbą sprawdzenia tej hipotezy dla wysoce symetrycznych konfiguracji punktów.

Jednym z celów rozprawy doktorskiej było obliczenie wartości stałej Waldschmidta dla niektórych konfiguracji punktów wyznaczonych przez zespolone grupy odbić. Postawiono hipotezę roboczą, że wartości te są wymierne i spełniają hipotezę Chudnowskiego. Obliczenia powinny być łatwiejsze niż w przypadku ogólnym ze względu na symetrie charakteryzujące układy odbić. Częściowe wyniki w tym kierunku uzyskano niedawno w [2]. Artykuł ten pokazuje jednak także, że w tej sytuacji potrzebne są nowe argumenty i pomysły. Na przykład istnieje jedynie oszacowanie stałej Waldschmidta układu Kleina (G_{30} w klasyfikacji Shepparda-Todda [27]) podane w [2]. Okazuje się, że dodatkowe problemy pojawiają się przy przejściu od konfiguracji na płaszczyźnie do konfiguracji w wyżej wymiarowych przestrzeniach rzutowych.

Kolejnym celem tej pracy było zebranie wiedzy, analiza metod, opracowanie nowych podejść i usystematyzowanie (lub uzupełnienie) wiedzy na temat asymptotycznych niezmienników. Zastosowano metody z algebry przemiennej i geometrii algebraicznej. Niektóre z nich zostały opracowane prez grupę działającą w Krakowie, do której należą m.in. Dumnicki, Malara, Szemberg, Szpond i Tutaj-Gasińska.

Niniejsza rozprawa złożona jest z pięciu rozdziałów.

W pierwszym rozdziale przedstawiamy kilka podstawowych definicji, przykładów i twierdzeń z geometrii algebraicznej i algebry przemiennej, które są niezbędne do dalszych rozważań.

W rozdziale drugim przypominamy pewne fakty dotyczące skończonych grup odbić. W tej części rozważymy własności konfiguracji D_4 (diagram Dynkina).

Trzecia część pracy doktorskiej polega na opisaniu niezmienników asymptotycznych, takich jak: stała Waldschmidta i resurgencja.

W rozdziale czwartym opisujemy stałą Waldschmidta symetrycznych układów punktów w przestrzeniach rzutowych oraz podajemy dowody teoretyczne.

W rozdziale piątym, przedstawiamy własności rzutów ogólnych symetrycznych zbiorów punktów w przestrzeniach rzutowych dla konfiguracji H_4.

Na końcu pracy, przedstawiamy kod programu w języku Singular [11] używany do wyliczenia stałej Waldschmidta dla rozważanych konfiguracji.

Niektóre dodatkowe krótkie kody programów są zawarte bezpośrednio w odpowiednich rozdziałach.

1 Preliminaries

The purpose of this chapter is to present some basic definitions, examples and theorems from algebraic geometry and commutative algebra.

Let $R = \mathbb{K}[x_0, \ldots, x_N]$ be the ring of polynomials over a field $\mathbb{K}$. Many statements made below are valid for arbitrary fields, some only for algebraically closed fields. In order to avoid confusion, we adopt the later assumption for the whole work.

The ring R has a natural graded structure

$$R = \bigoplus_{d \geqslant 0} R_d$$

given by degree of its homogeneous elements. For a given integer $k \geq 0$ we will denote by $R(-k)$ the ring R with grading shifted by k, i.e., $R(-k)_d = R_{d-k}$ for all $d \geq 0$.

The traditional objects of study are homogeneous ideals $\mathcal{I}$, which are algebraic objects and on the other hand, sets of zeros of systems of polynomial equations, which are geometric objects, denoted by $V(\mathcal{I})$. In our case, the ring of polynomials, which we consider, is Noetherian, so each algebraic set is described by a finite number of equations. When $\mathcal{I}$ is a principal ideal (generated by a unique polynomial), then we call $V(\mathcal{I})$ a hypersurface (the zero set of the generator of $\mathcal{I}$). If the generator has degree one, then we speak about a hyperplane.

Definition 1.1 (*Hyperplane*)
A *hyperplane* is a set of solutions to a linear equation in $\mathbb{P}^N$.

The same subset of the projective plane can be described by different ideals, for example if we consider the point $(0, 0)$ in the affine plane, then its maximal ideal is $\mathcal{I} = \langle x, y \rangle$, but $\mathcal{J} = \langle x^3, y^2 \rangle$ still describes the same point as a set, equipped however with a different algebraic structure. We are interested in the largest set of equations that describes a given set. This leads us to the definition of the *radical* of an ideal.

Definition 1.2 (*R*adical of an ideal)
Let $\mathcal{I}$ be an ideal of R. We define the *radical* $\sqrt{\mathcal{I}}$ *of* $\mathcal{I}$ as the set

$$\sqrt{\mathcal{I}} = \Big\{ r \in R : \quad \exists_{\, n \in \mathbb{N}_{\geqslant 1}} r^n \in \mathcal{I} \Big\}.$$

It is easy to see, that the radical of an ideal is also an ideal. We say that the ideal $\mathcal{I}$ is *radical* if $\mathcal{I} = \sqrt{\mathcal{I}}$.

Remark 1.3

For every ideal we have:

1. $\mathcal{I} \subset \sqrt{\mathcal{I}}$;

2. $\sqrt{\sqrt{\mathcal{I}}} = \sqrt{\mathcal{I}}$.

Thus taking the radical can be viewed as an algebraic operation which parallels taking the closure of a set in topology.

Now let us present an example.

Example 1.4

Let $\mathcal{I} = \langle x_0^2 + 3x_0x_1, 3x_0x_1 + x_1^2 \rangle \subseteq \mathbb{C}[x_0, x_1]$. Note that $\sqrt{\mathcal{I}} = \langle x_0, x_1 \rangle$ and we have $x_0 + x_1 \in \sqrt{\langle x_0^2 + 3x_0x_1, 3x_0x_1 + x_1^2 \rangle}$. Indeed

$$(x_0+x_1)^3 = x_0^3+3x_0^2x_1+3x_0x_1^2+x_1^3 = x_0(x_0^2+3x_0x_1)+x_1(x_1^2+3x_0x_1) \in \langle x_0^2+3x_0x_1, 3x_0x_1+x_1^2 \rangle.$$

Since every generator of $\mathcal{I}$ is of degree two and homogeneous, then

$$x_0 + x_1 \notin \langle x_0^2 + 3x_0x_1, 3x_0x_1 + x_1^2 \rangle.$$

It means that $\mathcal{I}$ is not a radical ideal.

Now we present a very important theorem which describes relationship between ideals and geometric objects [10].

Theorem 1.5 (*Nullstellensatz*)

Let $\mathcal{I} \subseteq \mathbb{K}[x_0, \ldots, x_N]$ be an ideal, where $\mathbb{K}$ is an algebraically closed field, then

$$\mathcal{I}(V(\mathcal{I})) = \sqrt{\mathcal{I}}.$$

Remark 1.6

We observe for further reference that

$$\mathcal{I}(X_1 \cup X_2) = \mathcal{I}(X_1) \cap \mathcal{I}(X_2).$$

Definition 1.7 (*Initial degree*)

For an arbitrary non-trivial homogeneous ideal $\mathcal{I} \subset R$ one defines its *initial degree* $\alpha(\mathcal{I})$ as the minimal d such that the homogeneous part $\mathcal{I}_d$ of degree d is non-zero. We have

$$\alpha(\mathcal{I}) := \min\{\deg f : 0 \not\equiv f \in \mathcal{I}\}.$$

The initial degree is an important invariant of an ideal. We introduce its asymptotic counterpart in Definition 3.1.

1.1 Symbolic powers of homogeneous ideals

Definition 1.8 (*Primary ideal*)
A proper ideal $Q \subset R$ is called *primary*, if every zero-divisor in the quotient ring R/Q is nilpotent.

Remark 1.9
The radical $P = \sqrt{Q}$ of a primary ideal Q is a prime ideal. We say that Q is P-primary.

An important result of Lasker and Noether identifies primary ideals as building blocks of all ideals in noetherian rings. More precisely we have the following theorem, see Chapter 3 in [15].

Theorem 1.10 (*Lasker-Noether*)
Any non-trivial homogeneous ideal $\mathcal{I} \subset R$ has a unique minimal (in the sense of inclusions) primary decomposition

$$\mathcal{I} = Q_1 \cap \cdots \cap Q_s,$$

where Q_i are primary ideals.

Definition 1.11 (*Associated primes*)
Let $\mathcal{I}$ be a proper ideal of a noetherian ring R and let

$$\mathcal{I} = Q_1 \cap \cdots \cap Q_s$$

be its minimal primary decomposition with $P_i = \sqrt{Q_i}$ for $i = 1, \ldots, s$. Then

$$\mathrm{Ass}(\mathcal{I}) = \{P_1, \ldots, P_s\}$$

is the set of *associated primes* of $\mathcal{I}$.

Definition 1.12 (*Symbolic power*)
Let $\mathcal{I} \subset R = \mathbb{K}[x_0, \ldots, x_N]$ be a homogeneous ideal. For a positive integer m, we define the m-th *symbolic power* of $\mathcal{I}$, as

$$\mathcal{I}^{(m)} = \bigcap_{p \in \mathrm{Ass}(\mathcal{I})} (\mathcal{I}^m R_p \cap R),$$

where $\mathrm{Ass}(\mathcal{I})$ is the set of associated primes of $\mathcal{I}$ and R_p is the localization of R at p.

In case $\mathbb{K}$ is an algebraically closed field symbolic power has a clear geometrical interpretation.

Theorem 1.13 (*Nagata-Zariski*)
Let $\mathcal{I} \subset \mathbb{K}[x_0, \dots, x_N]$ be a radical ideal and let $V(\mathcal{I})$ be the set of zeros. Then elements of $\mathcal{I}^{(m)}$ are all polynomials vanishing along $V(\mathcal{I})$ with multiplicity at least m.

In case of points, which are the main object of our interest, symbolic powers of their ideals have a particularly nice form.

Remark 1.14 (*Symbolic powers of points*)
Let $\mathcal{Z} = \{P_1, \dots, P_s\}$ be a finite set of points in $\mathbb{P}^N$. By Remark 1.6 we have

$$\mathcal{I}(\mathcal{Z}) = \mathcal{I}(P_1) \cap \dots \cap \mathcal{I}(P_s),$$

where $\mathcal{I}(P_i)$ is the ideal of polynomials which vanish at the point P_i.

For $m \geqslant 1$, by Theorem 1.13, the symbolic power $\mathcal{I}(\mathcal{Z})^{(m)}$ of the ideal $\mathcal{I}(\mathcal{Z})$ is computed as

$$\mathcal{I}(\mathcal{Z})^{(m)} = \mathcal{I}(P_1)^m \cap \dots \cap \mathcal{I}(P_s)^m.$$

Example 1.15
Consider 3 non-collinear points in $\mathbb{P}^2$. Without loss of generality we can assume that their coordinates are:

$$P_1 = [1 : 0 : 0]\,, \qquad P_2 = [0 : 1 : 0]\,, \qquad P_3 = [0 : 0 : 1]\,.$$

Then for:

1. $m = 1$, we obtain $\mathcal{I} = \mathcal{I}(P_1) \cap \mathcal{I}(P_2) \cap \mathcal{I}(P_3) = \langle y, z\rangle \cap \langle x, z\rangle \cap \langle x, y\rangle = \langle yz, xz, xy\rangle$;
2. $m = 2$, we obtain $\mathcal{I}^{(2)} = \mathcal{I}(P_1)^2 \cap \mathcal{I}(P_2)^2 \cap \mathcal{I}(P_3)^2 = \langle y, z\rangle^2 \cap \langle x, z\rangle^2 \cap \langle x, y\rangle^2 = \langle z^2, yz, y^2\rangle \cap \langle z^2, xz, x^2\rangle \cap \langle y^2, xy, x^2\rangle = \langle xyz, x^2y^2, x^2z^2, y^2z^2\rangle$;
3. $m = 3$, we obtain $\mathcal{I}^{(3)} = \mathcal{I}(P_1)^3 \cap \mathcal{I}(P_2)^3 \cap \mathcal{I}(P_3)^3 =$
$= \langle xy^2z^2, x^2yz^2, x^2y^2z, y^3z^3, x^3z^3, x^3y^3\rangle$.

Definition 1.16 (*Fat point*)
For $m \geq 2$, a *fat point* mP is a scheme supported on a point P, whose structure is defined by $\mathcal{I}(P)^m$.

Definition 1.17 (*Fat point scheme*)
Let $m_1, \ldots, m_s$ be positive integers. We say that $Z = m_1 P_1 + \cdots + m_s P_s$ is a *fat points scheme* its ideal is

$$\mathcal{I}(Z) = \bigcap_{i=1}^{s} \mathcal{I}(P_i)^{m_i}.$$

For fat point schemes there is a statement which parallels Remark 1.14.

Remark 1.18 (*Symbolic powers of fat point schemes*)
Let $\mathcal{I}(\mathcal{Z}) = \mathcal{I}(P_1)^{m_1} \cap \cdots \cap \mathcal{I}(P_s)^{m_s}$, then the $m-$th symbolic power of $\mathcal{I}(\mathcal{Z})$ is given by:

$$\mathcal{I}(\mathcal{Z})^{(m)} = \mathcal{I}(P_1)^{mm_1} \cap \cdots \cap \mathcal{I}(P_s)^{mm_s}.$$

1.2 Resolution of homogeneous ideals

We assume here for simplicity that R is the ring of complex polynomials and all ideals studied here are homogeneous.

The ring of polynomials is noetherian (by Hilbert's basis theorem), i.e., any ideal $\mathcal{I} \subset R$ is finitely generated. This property is equivalent to the existence of a surjective morphism of graded R-modules

$$\bigoplus_{i=1}^{k} R(-a_i) \to \mathcal{I} \to 0.$$

It is an intriguing question to find the simplest set of generators and also the simplest set of relations between them (syzygies). The simplicity is measured by the cardinality of the set of generators.

Definition 1.19 (*Minimal set of generators of $\mathcal{I}$*)
We say that a set $\{g_1, g_2, \ldots, g_k\}$ is a *minimal set of generators of $\mathcal{I} \subset R$* if this set generates $\mathcal{I}$ and no proper subset of this set has this property.

Generators of an ideal deliver, in principle, all information about the structure of the ideal but it is somehow concealed by relations between them. Around 1890 Hilbert, building upon earlier works of Cayley, put forward the concept of free resolutions. Before we define it we need to make a convention.

Notation 1.20
To fix the notation we adopt convention that a matrix operates on a vector (represented by a column of coordinates) by multiplying from the left as in the example below. For

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \text{ and } v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \text{ we have}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} a_{11} \cdot v_1 + a_{12} \cdot v_2 + a_{13} \cdot v_3 \\ a_{21} \cdot v_1 + a_{22} \cdot v_2 + a_{23} \cdot v_3 \end{bmatrix}.$$

Definition 1.21 (*Free resolution*)
Let $\mathcal{I} \subset R$ be a homogeneous ideal. A *free resolution* of $\mathcal{I}$ is an exact sequence of the form

$$0 \to \bigoplus_{m_n} R(-a_{nm_n}) \xrightarrow{A_n} \dots \xrightarrow{A_2} \bigoplus_{m_1} R(-a_{1m_1}) \xrightarrow{A_1} \bigoplus_{m_0} R(-a_{0m_0}) \to \mathcal{I} \to 0,$$

where A_i are matrices whose entries are elements of R.

The finiteness of the resolution follows from Hilbert's Syzygy Theorem. A proper ideal has many different free resolutions. Among them one can distinguish the minimal ones.

Definition 1.22 (*Minimal free resolution of* $\mathcal{I}$)
We say that a graded free resolution is a *minimal graded free resolution of* $\mathcal{I}$ if no elements invertible in R (non-zero scalars in our case) appear as entries in matrices A_i.

There is an important invariant associated to free resolutions.

Definition 1.23 (*Castelnuovo-Mumford regularity*)
Let $\mathcal{I} \subset R$ be an ideal $\mathcal{I}$ with a minimal free resolution given by

$$0 \to \bigoplus_{m_n} R(-a_{nm_n}) \to \dots \to \bigoplus_{m_1} R(-a_{1m_1}) \to \bigoplus_{m_0} R(-a_{0m_0}) \to \mathcal{I} \to 0.$$

The *Castelnuovo-Mumford regularity* $\operatorname{reg}(\mathcal{I})$ *of an ideal* $\mathcal{I}$ is the number

$$\operatorname{reg}(\mathcal{I}) := \max_{k,m}\{a_{km} - k\}.$$

Remark 1.24
The number $\operatorname{reg}(\mathcal{I})$ can be considered as a way to measure the complication of the ideal. Note that $\operatorname{reg}(\mathcal{I})$ is at least equal to the highest degree generator of $\mathcal{I}$ in a minimal set of generators.

If $\mathcal{I}$ is generated in a single degree d and the regularity is d, then we speak about a *linear resolution* of $\mathcal{I}$. It is the simplest situation for non-trivial ideals; all entries in matrices A_i are either 0 or linear forms.

Example 1.25

Considering the points

$$P_1 = [1:0:0]\,, \qquad P_2 = [0:1:0]\,, \qquad P_3 = [0:0:1]\,.$$

from the previous Example 1.15. Then for

1. $m = 1$ and $\mathcal{I} = \langle yz, xz, xy \rangle$ the minimal free resolution is

$$0 \to R^2(-3) \xrightarrow{\begin{bmatrix} -x & y & 0 \\ -x & 0 & z \end{bmatrix}^T} R^3(-2) \xrightarrow{\begin{bmatrix} yz & xz & xy \end{bmatrix}} \mathcal{I} \to 0.$$

   This implies

$$\operatorname{reg}(\mathcal{I}) = \max\{2-0, 3-1\} = 2.$$

2. $m = 2$ and $\mathcal{I}^{(2)} = \langle xyz, x^2y^2, x^2z^2, y^2z^2 \rangle$ the minimal free resolution is

$$0 \to R^3(-5) \xrightarrow{\begin{bmatrix} -yz & x & 0 & 0 \\ -xz & 0 & y & 0 \\ -xy & 0 & 0 & z \end{bmatrix}^T} R(-3) \oplus R^3(-4) \xrightarrow{\begin{bmatrix} xyz & x^2y^2 & x^2z^2 & y^2z^2 \end{bmatrix}} \mathcal{I} \to 0.$$

   This implies

$$\operatorname{reg}(\mathcal{I}^{(2)}) = \max\{3-0, 4-0, 5-1\} = 4.$$

3. $m = 3$ and $\mathcal{I}^{(3)} = \langle xy^2z^2, x^2yz^2, x^2y^2z, y^3z^3, x^3z^3, x^3y^3 \rangle$ the minimal free resolution is

$$0 \to R^2(-6) \oplus R^3(-7) \xrightarrow{\begin{bmatrix} -x & y & 0 & 0 & 0 & 0 \\ -x & 0 & z & 0 & 0 & 0 \\ -yz & 0 & 0 & x & 0 & 0 \\ 0 & -xz & 0 & 0 & y & 0 \\ 0 & 0 & -xy & 0 & 0 & z \end{bmatrix}^T}$$

$$R^3(-5) \oplus R^3(-6) \xrightarrow{\begin{bmatrix} xy^2z^2 & x^2yz^2 & x^2y^2z & y^3z^3 & x^3z^3 & x^3y^3 \end{bmatrix}} \mathcal{I} \to 0.$$

   This implies

$$\operatorname{reg}(\mathcal{I}^{(3)}) = \max\{5-0, 6-0, 6-1, 7-1\} = 6.$$

We enclose here a short program in Singular [11], which can be used to verify quickly above claims.

```
LIB "primdec.lib";
option(redSB);
ring r=0,(x,y,z),dp;

proc function_of_Hilbert(poly a, poly b, poly c){
   maxideal(3);
   ideal I = a,b,c;
   return(size(NF(maxideal(5),std(I))));
}
ideal I;
I= y*z, x*z , x*y;
print("Regularity: " + string(regularity(mres(I, 0))));

resolution rs= res(I,0);
rs;
print(betti(rs), "betti");

print(matrix(rs[2]));
```

Output:

Regularity: 2

$$\begin{array}{ccccc} 1 & & 3 & & 2 \\ r & <-- & r & <-- & r \\ 0 & & 1 & & 2 \end{array}, \qquad \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0: & 1 & - & - \\ 1: & - & 3 & 2 \\ \textit{total}: & 1 & 3 & 2 \end{array}, \qquad \begin{bmatrix} -x & -x \\ y & 0 \\ 0 & z \end{bmatrix}.$$

1.3 Intersection theory and the Theorem of Bezout

In this section we recall basic notions and properties relevant in the subsequent parts of this work.

Definition 1.26 (***Multiplicity at a point***)

We say that a homogeneous polynomial $f(x_0, x_1, \dots, x_N)$ vanishes to order k at a point

$P = (p_0 : p_1 : \cdots : p_N)$, if

$$\frac{\partial^{k-1} f}{\partial x_0^{i_0} \partial x_1^{i_1} \ldots \partial x_N^{i_N}} (p_0 : p_1 : \cdots : p_N) = 0,$$

for all $i_0, \ldots, i_N$ such that $i_0 + i_1 + \cdots + i_N = k - 1$, but for some $a_0, a_1, \ldots, a_N \in \mathbb{N}$ such as $a_0 + a_1 + \cdots + a_N = k$

$$\frac{\partial^{k} f}{\partial x_0^{a_0} \partial x_1^{a_1} \cdots \partial x_N^{a_N}} (p_0 : p_1 : \cdots : p_N) \neq 0.$$

We denote the multiplicity of f at the point P by $m_P(f)$.

If $\mathcal{H}$ is the hypersurface defined by the vanishing of f, by a slight abuse of notation, we denote its multiplicity at a point P by $m_P(\mathcal{H})$ rather than $m_P(f)$. This is convenient if we have a hypersurface and we are not interested in its equation. In the next Definition we actually use the convenience of denoting a hypersurface (a curve in that instance) and its equation by the same letter.

Next we recall the definition of the local intersection of two algebraic curves. This is a central object in our applications, see [19].

Definition 1.27 (*Intersection numbers of curves $\mathcal{C}$ and $\mathcal{D}$*)
For all curves $\mathcal{C}$ and $\mathcal{D}$ in $\mathbb{P}^2$ and all points $P \in \mathbb{P}^2$ We define the local intersection numbers of *calc* and $\mathcal{D}$ at P as numbers $I(P, \mathcal{C} \cap \mathcal{D})$ (possible equal to infinity), subject to the following conditions:

1. If curves $\mathcal{C}$ and $\mathcal{D}$ have no common components at P then $I(P, \mathcal{C} \cap \mathcal{D})0$ is a non-negative integer. Otherwise $I(P, \mathcal{C} \cap \mathcal{D}) = \infty$.

2. A point $P \notin \mathcal{C} \cap \mathcal{D}$ if and only if $I(P, \mathcal{C} \cap \mathcal{D}) = 0$.

3. If T is a linear change of coordinates in $\mathbb{P}^2$, then

$$I(P, \mathcal{C} \cap \mathcal{D}) = I(T(P), T(\mathcal{C}) \cap T(\mathcal{D})).$$

4. $I(P, \mathcal{C} \cap \mathcal{D}) = I(P, \mathcal{D} \cap \mathcal{C})$.

5. There is $I(P, \mathcal{C} \cap \mathcal{D}) \geqslant m_P(\mathcal{C}) \cdot m_P(\mathcal{D})$. The equality holds if and only if the curves $\mathcal{C}$ and $\mathcal{D}$ have no common tangent line at the point P.

6. Let $\mathcal{C}_i$ and $\mathcal{D}_j$ be polynomials such that $\mathcal{C} = \prod_i \mathcal{C}_i^{r_i}$ and $\mathcal{D} = \prod_j \mathcal{D}_j^{s_j}$, then we have

$$I(P, \mathcal{C} \cap \mathcal{D}) = I(P, \prod_i \mathcal{C}_i^{r_i} \cap \prod_j \mathcal{D}_j^{s_j}) = \sum_{i,j} r_i s_j I(P, \mathcal{C}_i \cap \mathcal{D}_j).$$

7. $I(P, \mathcal{C} \cap \mathcal{D})) = I(P, \mathcal{C} \cap (\mathcal{D} + f \cdot \mathcal{C}))$, for any $f \in \mathbb{K}[x_0, x_1, x_2]$.

Theorem 1.28 (*Fulton*)

There exist unique numbers $I(P, \mathcal{C} \cap \mathcal{D})$ satisfying all conditions in Definition 1.27.

PROOF. For a proof we refer to [19] (Section 3.3, Theorem 3). □

Two comments are in place. First we address the computability of the introduced invariants.

Remark 1.29

Definition 1.27 and Theorem 1.28 secure the *existence* of the local intersection numbers but are not constructive in the sense that they don't provide a direct way of computing these numbers. To this end one can use *resultants* as described in [21].

The next comment concerns local intersection numbers between higher dimensional objects of complementary dimensions. They can be defined in a similar manner and a generalized Bezout's theorem holds also in this case, see [18] for details.

Rather than dwelling on the general intersection theory let us present the most important for us result involving local intersection numbers.

Theorem 1.30 (*Bezout*)

Let $\mathcal{C}$ and $\mathcal{D}$ be curves in $\mathbb{P}^2(\mathbb{C})$ of degree m and n respectively. Assume that $\mathcal{C}$ and $\mathcal{D}$ have no common components. Then

$$(\deg(\mathcal{C}))(\deg(\mathcal{D})) = \sum_{P \in \mathcal{C} \cap \mathcal{D}} I(P, \mathcal{C} \cap \mathcal{D}) = m \cdot n.$$

PROOF. See [19] (Section 5.3). □

Example 1.31

Let $\mathcal{L}$ be a line and $\mathcal{C}$ a conic defined over an algebraically closed field. There are two situations possible:

a) $\mathcal{L} \cap \mathcal{C} = \{P\}$,

b) $\mathcal{L} \cap \mathcal{C} = \{Q, R\}$.

Then it is $\mathcal{I}(P, \mathcal{L} \cap \mathcal{C}) = 2$ in a), whereas $\mathcal{I}(Q, \mathcal{L} \cap \mathcal{C}) = \mathcal{I}(R, \mathcal{L} \cap \mathcal{C}) = 1$ in b). This is illustrated in the next two figures.

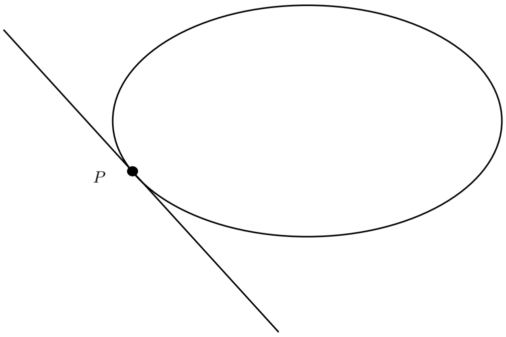


Figure 1.1: a) one point of multiplicity 2

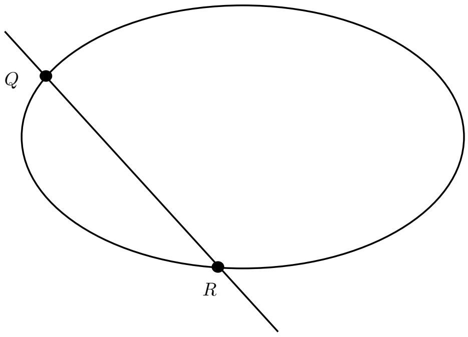


Figure 1.2: b) two points of multiplicity 1

2 Complex reflection groups

In this section we will recall some basic facts about arrangements associated to finite reflection groups.

Definition 2.1 (*Group action*)
Let X be a nonempty set and let $(G, \star)$ be a group. A (left) group action of G on X is a map $G \times X \to X$, given by $(g, x) \mapsto g \cdot x$ such that

1. for the neutral element $e \in G$ there is $e \cdot x = x$ for all $x \in X$;

2. $(g_1 \star g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$ for all $x \in X$ and $g_1, g_2 \in G$.

Definition 2.2 (*Orbit*)
The *orbit* of an element $x \in X$ is the set of all these elements $y \in X$ such that $y = gx$ for some $g \in G$. We write $O(x) = \{y \in X : y = g \cdot x\} = \{g \cdot x, g \in G\}$.

Definition 2.3 (*Fixed point*)
We say point $x \in X$ is a *fixed point* of $f : X \to X$, if

$$f(x) = x.$$

More generally x is fixed point of a group action if $f(x) = x$ for all $f \in G$. In particular a fixed point has an orbit consisting just of that point.

We denote the set of fixed points of f by $\mathrm{Fix}(f)$.

In this part we consider only $\mathbb{K} = \mathbb{C}$ so here $R = \mathbb{C}[x_0, \ldots, x_N]$. Let V be the vector space $\mathbb{C}^{N+1}$ with coordinates $x_0, \ldots, x_N$.

Example 2.4
For R and a finite group G of linear automorphisms of the vector space V we have that G acts on R so that $g \cdot f$ is defined by

$$(g \cdot f)(x) = f(g^{-1}(x)), \text{ for all } x \in V,$$

where $f \in R$ and $g \in G$.

Definition 2.5 (***Invariant polynomial***)
Let V be the vector space as above and let G be a finite subgroup of $\mathrm{GL}(V)$ acting on the ring of polynomials R. We say that $f \in R$ is an *invariant polynomial* with respect to the action of G if

$$(g \cdot f)(x) = f(x),$$

for all $g \in G$ and $x \in V$.

In other words, an invariant polynomial is a fixed point of the action of G on R.

Example 2.6
Let $G = \{id, -id\}$, then G acts on R in the following way

1. for $g = id$ this is a trivial case, $g \cdot f = f$ for all $f \in R$;
2. for $g = -id$ we have $(g \cdot f)(x) = f(-x)$.

Thus a general polynomial $f \in R$ has an orbit consisting of 2 elements: $f(x), f(-x)$. But an even polynomial, i.e., a polynomial f such the $f(x) = f(-x)$ for all $x \in V$, has an orbit consisting only of one element. So, in particular, in this case every even polynomial is invariant under G and vice versa.

Definition 2.7 (***Order of a matrix***)
We say that a matrix A has order k, if k is the minimal number with the property

$$A^k = id.$$

Thus if a matrix has an order then it is in particular invertible. Of course there are invertible matrices which don't have a finite order.

Example 2.8
Let $R = \mathbb{C}[x_0, \dots, x_N]$ and $G = \{id, g'\} \subset \mathrm{GL}(V)$, where g' is defined as follows

$$g'(x_0, x_1, \dots, x_N) = (-x_0, x_1, \dots, x_N).$$

The automorphism g' is represented by the following associated $(N+1) \times (N+1)$ matrix

$$\begin{pmatrix} -1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

From the above form of the matrix we can see that g' has order 2 and exactly N eigenvalues equal to 1.

Consider symmetric polynomials

$$f_1 := x_0 \cdots x_N, \quad f_2 := x_0 + \cdots + x_N.$$

Then

- $O(f_1) = \{f \in R : f = gf_1 \text{ for } g \in G\} = \{f_1, -f_1\}$,
- $O(f_2) = \{x_0 + \cdots + x_N, -x_0 + \cdots + x_N\}$.

Invariant polynomials $f \in R$ are those, which belong to $\mathbb{C}[x_0^2, x_1, \ldots, x_N]$.

In this part, we define reflections and reflection groups based on [24].

Definition 2.9 (*Reflection*)
A *reflection* in $\mathbb{C}^{N+1}$ is a linear automorphism of finite order which has exactly N eigenvalues equal to 1.

The finite order of a reflection forces its remaining eigenvalue to be a root of unity.

Definition 2.10 (*Reflection group*)
We say that $G \subseteq \mathrm{GL}(V)$ is a *reflection group* if it is generated by reflections.

In the historical context the complex reflection groups were considered as the first.

Remark 2.11
If s is a reflection, then the set $\mathrm{Fix}(s)$ is a hyperplane, called the reflecting hyperplane of s.

Example 2.12
Define a set of automorphisms of V, by:

$$s_i : (x_0, x_1, \ldots, x_N) \longmapsto (\varepsilon_k^i \cdot x_0, x_1, \ldots, x_N),$$

where ε_k is a primitive root of unity of order k, then $\{s_i\}_{i \in \{1,\ldots,k\}}$ form a reflection group. It is in this case a cyclic group of order k.

Remark 2.13
Given a reflection of order $k \geq 2$, by [24] there exists a primitive root of unity ε of order k and a point $a \in V$ such that $s = s_{a,\varepsilon}$ and

$$s_{a,\varepsilon}(x) = x - (1 - \varepsilon) \frac{< x, a >}{< a, a >} a.$$

The next two definitions are based on [24].

Definition 2.14 (*Reflection arrangement*)

A *reflection arrangement* is a hyperplane arrangement $\mathcal{H}(G)$ which consists of reflection hyperplanes defined by reflections in a finite reflection group G.

Definition 2.15 (*Irreducible group*)

A complex reflection group $G \subset \mathrm{GL}(V)$ is called *irreducible* if there is no non-trivial invariant proper subspace of V invariant under G.

Example 2.16 (*Monomial groups*)

Let $\Sigma_{N+1} \subset \mathrm{GL}(V)$ be the group of all $(N+1) \times (N+1)$ permutation matrices. It is of course isomorphic with the permutation group S_{N+1} of $(N+1)$ elements. Let $n \geq 2$ and $p \geq 1$ be integers with $p|n$ and let $A(n, p, N+1)$ be the group of $(N+1) \times (N+1)$ diagonal matrices $A = (a_{ij})_{i,j\in[N+1]}$ with $a_{ij} = \varepsilon^{\alpha_i}\delta_{ij}$, where ε is a primitive root of unity of order n, $\alpha_i \in \{1, ..., n\}$ and such the product

$$\det(A) = \prod_{i\in[N+1]} a_{ii}$$

is a power of ε^p. Let $G(n, p, N+1)$ be the semi-direct product of $A(n, p, N+1)$ and Π_{N+1}. Then $G(n, p, N+1)$ is an irreducible complex reflection group.

In the mid of 20th century, Shephard and Todd [27] classified finite complex reflection groups. In brief they found that every irreducible complex reflection group is either one of $G(n, p, N+1)$ (denoted in the literature also as $G(m, p, n)$) or one of 34 exceptional examples denoted traditionally by G_i with $i = 4, \ldots, 37$.

Example 2.17

Consider 3 non-collinear points in $\mathbb{P}^2$:

$$P_1 = [1:0:0]\,, \qquad P_2 = [0:1:0]\,, \qquad P_3 = [0:0:1]\,.$$

This situation is visualized at figure:

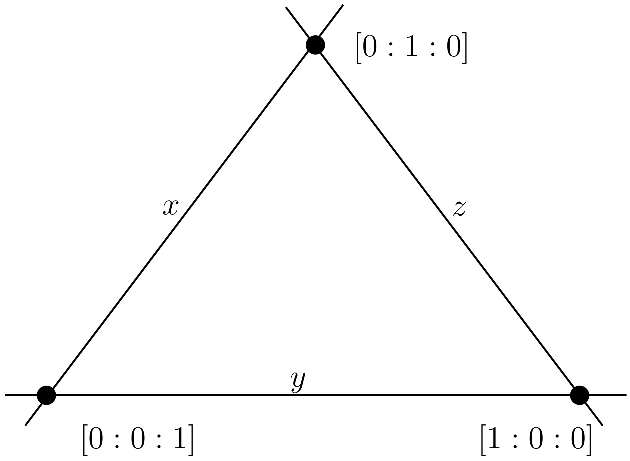


Figure 2.1: Configuration of 3 non-collinear in $\mathbb{P}^2$

Let us begin with an example of the reflection S_y, which corresponds to the hyperplane $x - z = 0$. As a map we have $S_y : [x : y : z] \longrightarrow [z : y : x]$.

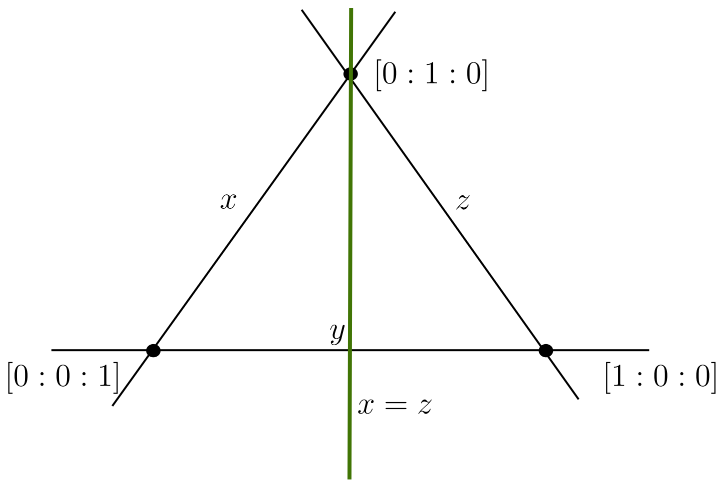


Figure 2.2: Reflection

When we consider S_z, which corresponds to the hyperplane $y - x = 0$, we obtain $S_z : [x : y : z] \longrightarrow [y : x : z]$. And the last reflection is S_x, which corresponds to equation $y - z = 0$. We obtain that $S_x : [x : y : z] \longrightarrow [x : z : y]$.

If we consider $S_y \circ S_y$, then we get identity. The same holds for $S_x \circ S_x$ and for $S_z \circ S_z$.

We can take composition of two different reflections. For example $S_a = S_x \circ S_y$ then we obtain that $[x : y : z] \longrightarrow [z : y : x] \longrightarrow [z : x : y]$. This transformation is represented by the matrix

$$S_a = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

It is easy to see that the eigenvalues of this matrix are $1, \frac{-1+\sqrt{3}i}{2}$ and $\frac{-1-\sqrt{3}i}{2}$, so that in particular this transformation is not a reflection.

Similarly, for $S_b = S_y \circ S_x$, then we obtain that $[x : y : z] \longrightarrow [x : z : y] \longrightarrow [y : z : x]$. So that in particular $S_y \circ S_x \neq S_x \circ S_y$.

Taking $S_y \circ S_x \circ S_y$ we get

$$[x : y : z] \longrightarrow [z : y : x] \longrightarrow [z : x : y] \longrightarrow [y : x : z],$$

but it is S_z.

Finally, when we consider all possible compositions, we obtain 6 transformation: the 3 reflections: S_x, S_y, S_z and the three non-reflections $\mathrm{Id}, S_a : [x : y : z] \longrightarrow [z : x : y]$ and $S_b : [x : y : z] \longrightarrow [y : z : x]$.

We enclose here a program that calculates the group generated by S_x, S_y and S_z using the Reynolds operator.

```
LIB "finvar.lib";
ring R=0,(x,y,z),dp;
//S_x
matrix A[3][3]=1,0,0, 0,0,1, 0,1,0;
//S_y
matrix B[3][3]=0,0,1, 0,1,0, 1,0,0;
//S_z
matrix C[3][3]=0,1,0, 1,0,0, 0,0,1;
list L=group_reynolds(A,B,C);
print(L[1]);
```

Output:

$S_x : [x \;\; z \;\; y]$, $S_y : [z \;\; y \;\; x]$, $S_z : [y \;\; x \;\; z]$, $id : [x \;\; y \;\; z]$, $S_a : [y \;\; z \;\; x]$, $S_b : [z \;\; x \;\; y]$. These 6 elements form the permutation group S_3.

2.1 Root systems

We present here some necessary definitions for further consideration in this and the following chapters, consult [9] and also [28] for more details.

Definition 2.18 (*Root system*)
A *root system* Δ is a finite set of vectors in an affine space V (in our case it will be $\mathbb{R}^3$) such that

- the elements in Δ span V;
- for each $a \in \Delta$, the vector $-a$ is also in Δ and no other multiple of a is there;
- for each a and b in Δ, the vector $s_a(b)$ is also in Δ (here $s_a(\upsilon) = \upsilon - 2\frac{\langle a,\upsilon\rangle}{\langle a,a\rangle}a$ is the image of υ under the reflection in the hyperplane perpendicular to a);
- for each a and b in Δ, the number $2\,\frac{\langle a,b\rangle}{\langle a,a\rangle}$ is an integer.

Each element of Δ is called a *root*.

Definition 2.19 (*Simple roots*)
Let Δ be a root system in an affine space V, then $\Sigma \subset \Delta$ is a *fundamental system of* Δ if it is linearly independent and every element of Δ can be written as a linear combination of elements of Σ such that all coefficients are either nonnegative or nonpositive.
The elements of Σ are called *simple roots*.

Theorem 2.20 (*Existence of a fundamental system*)
Every root system has a fundamental system.

Definition 2.21 (*Equivalence of root systems*)
Let $\Delta \subset V$ and $\Delta' \subset V'$ be root systems. We say that Δ and Δ' are isomorphic if there exists a linear map $f : V \to V'$, satisfying the following conditions:

- $f(\Delta) = \Delta'$;
- $\forall a, b \in \Delta$ we have $\langle f(a), f(b)\rangle = \langle a, b\rangle$.

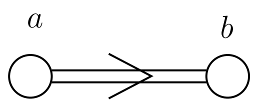
Now we pass to the construction of a graph theoretical object called a *Dynkin diagram* associated to a root system.

Definition 2.22 (*Dynkin diagram*)
A Dynkin diagram of a root systen Δ is a graph constructed according to the following rules:

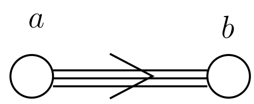
- we choose a set of simple roots Σ from Δ, which are then in a $1:1$ correspondence with the vertices of the graph;
- to any two distinct roots $a \neq b \in \Sigma$ we assign $0, 1, 2$ or 3 edges between the graph vertices corresponding to them depending on the angles (in the affine space) between the roots:

$0 \quad 0$	$\frac{\pi}{2}$	If the angle between two roots is equal to $\frac{\pi}{2}$, we don't join vertices by an edge.
$0 - 0$	$\frac{2\pi}{3}$	If the angle between two roots is equal to $\frac{2\pi}{3}$, we join vertices by one edge.
$0 = 0$	$\frac{3\pi}{4}$	If the angle between two roots is equal to $\frac{3\pi}{4}$, we join vertices by two edges.
$0 \equiv 0$	$\frac{5\pi}{6}$	If the angle between two roots is equal to $\frac{5\pi}{6}$, we join vertices by three edges.

- if there is an edge between two vertices a and b, we consider additionally the length of the corresponding roots and decorate the edge with an arrow pointing from a to b iff $\|a\| > \|b\|$. For example:



or



2.2 The D_4 Configuration

Now we are in the position to study a first root system and its associated points in the projective space in more detail.

Definition 2.23

The D_4 root system consists of 24 roots in $\mathbb{R}^4$. Taking their images in the projective space the roots a and $-a$ get identified. So we get only 12 points in $\mathbb{P}^3$. We choose their coordinates as:

$$P_1 = [1:-1:0:0]\,, \qquad P_2 = [0:1:-1:0]\,, \qquad P_3 = [0:0:1:-1]\,,$$
$$P_4 = [0:0:1:1]\,, \qquad P_5 = [1:0:-1:0]\,, \qquad P_6 = [0:1:0:-1]\,,$$
$$P_7 = [0:1:0:1]\,, \qquad P_8 = [1:0:0:-1]\,, \qquad P_9 = [1:0:0:1]\,,$$
$$P_{10} = [0:1:1:0]\,, \qquad P_{11} = [1:0:1:0]\,, \qquad P_{12} = [1:1:0:0]\,.$$

The first four elements of the above set can be taken as simple roots. The length of each vector is equal to $\sqrt{2}$, so there will be no arrows in the Dynkin diagram. Checking angles between these points (viewed as vectors in the affine space) we get the following pairs of orthogonal vectors:

$$P_1 \perp P_4, \quad P_1 \perp P_3, \quad P_3 \perp P_4.$$

For the remaining pairs of vectors we obtain always the angle $\frac{\pi}{3}$.

This data suffices to construct the Dynkin diagram, which is visualized in Figure 2.3.

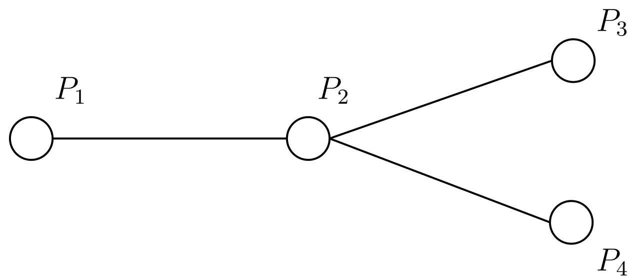


Figure 2.3: Dynkin diagram of the D_4 configuration

The group generated by reflections for the above configuration has order $192 = 8 \cdot 4!$ [9].

Definition 2.24 (*Representation of D_4 as a subgroup of* $\mathbb{P}\,\mathrm{GL}(4)$)
The following 4 matrices generate the reflection group D_4.

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad A_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

If we apply any of the above matrices to a point in the D_4 configuration, we get either the same point, or some other point of the configuration.

3 Asymptotic invariants

Recent decades have witnessed a notable shift in focus, moving from the study of isolated objects to examining families of objects and their limit behavior. This shift has been observed in both algebraic geometry and commutative algebra. In the theory of linear series, significant pioneering work has been conducted, with notable contributions by Fujita [17]. To understand and quantify the asymptotic properties of various algebraic objects—often inspired by geometric ideas—several invariants have been introduced. In this section, we will discuss two of these asymptotic invariants.

3.1 Waldschmidt constants

In this part of the chapter, we introduce one of the asymptotic invariant derived from the initial degree ($\alpha(\mathcal{I})$) of the ideal and its symbolic powers (see Definition 1.7).

Definition 3.1 (*Waldschmidt constant*)
Let $\mathcal{I}$ be a nontrivial homogeneous ideal in R. The *Waldschmidt constant* of $\mathcal{I}$ is the real number

$$\widehat{\alpha}(\mathcal{I}) = \lim_{m\to\infty} \frac{\alpha(\mathcal{I}^{(m)})}{m},$$

where $\mathcal{I}^{(m)}$ is the m-th symbolic power of $\mathcal{I}$.

Lemma 3.2 (*Subadditivity of the initial degree of symbolic powers of an ideal*)
Let $\mathcal{I}$ be a radical homogeneous ideal in $R = \mathbb{K}[x_0, \ldots, x_N]$. Then

$$\alpha(\mathcal{I}^{(k+\ell)}) \leq \alpha(\mathcal{I}^{(k)}) + \alpha(\mathcal{I}^{(\ell)})$$

for all positive $k, \ell \in \mathbb{Z}$.

PROOF. By the Nagata-Zariski Theorem 1.13 there is

$$\mathcal{I}^{(k)}\mathcal{I}^{(\ell)} \subset \mathcal{I}^{(k+\ell)}$$

and the assertion follows. □

It is natural to wonder if there is actually an equality in Lemma 3.2. The next example shows that even for a very simple set of points and various values of powers k and ℓ we can obtain either equality or strict inequality. Turning to the details we revoke Example 2.17 from chapter two. For reader's convenience we repeat it here.

Example 3.3

Consider 3 non-collinear points in $\mathbb{P}^2$:

$$P_1 = [1:0:0], \qquad P_2 = [0:1:0], \qquad P_3 = [0:0:1].$$

This configuration is visualised at Figure 3.1.

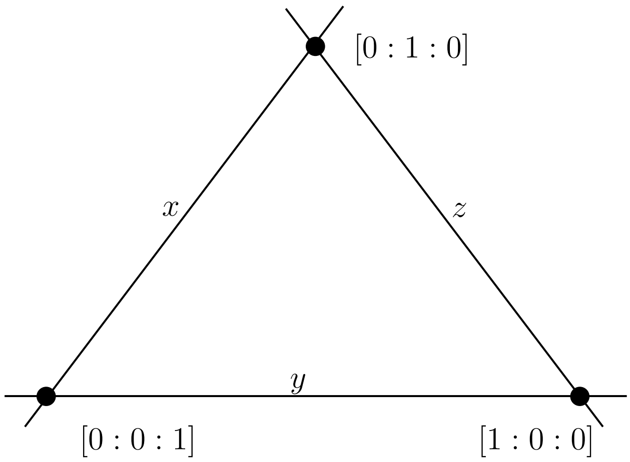


Figure 3.1: Configuration of 3 non-collinear in $\mathbb{P}^2$

From Example 1.25, we have for $k = 1$ that

$$\mathcal{I}^{(1)} = \mathcal{I}(P_1) \cap \mathcal{I}(P_2) \cap \mathcal{I}(P_3) = \langle xy, xz, yz \rangle$$

and $\alpha(\mathcal{I}) = 2$.

In turn, for $\ell = 3$ we get

$$\mathcal{I}^{(3)} = \langle xy^2z^2, x^2yz^2, x^2y^2z, y^3z^3, x^3z^3, x^3y^3 \rangle$$

and $\alpha(\mathcal{I}^{(3)}) = 5$. As a consequence there is a strict inequality in Lemma 3.2 because

$$\alpha(\mathcal{I}^{(1)}) + \alpha(\mathcal{I}^{(3)}) = 7$$

but

$$\alpha(\mathcal{I}^{(1+3)}) = 6,$$

as $x^2y^2z^2 \in \mathcal{I}^{(4)}$. Thus we obtain

$$6 = \alpha(\mathcal{I}^{(1+3)}) < \alpha(\mathcal{I}^{(1)}) + \alpha(\mathcal{I}^{(3)}) = 7.$$

On the other hand, for $\ell = 2$ we computed in Example 1.25

$$\mathcal{I}^{(2)} = \langle xyz, y^2z^2, x^2z^2, x^2y^2 \rangle,$$

so that $\alpha(\mathcal{I}^{(2)}) = 3$.

Then in Lemma 3.2 there is equality because

$$\alpha(\mathcal{I}^{(1)}) + \alpha(\mathcal{I}^{(2)}) = 5$$

and also $\alpha(\mathcal{I}^{(1+2)}) = 5$.

Applying Fekete's Lemma from [16], we conclude that any subadditive sequence of real numbers is bounded from below and converges to the infimum of its terms. This, in turn, implies both the existence of the limit in Definition 3.1 and the equality:

$$\widehat{\alpha}(\mathcal{I}) = \inf \frac{\alpha(\mathcal{I}^{(m)})}{m}.$$

Waldschmidt constants were recently rediscovered and studied by Bocci and Harbourne in [6]. They used the symbol $\gamma(\mathcal{I})$. But in [4] the authors proposed the notation $\widehat{\alpha}(\mathcal{I})$, which became standard.

Waldschmidt constants are very difficult to determine in general. Suffices it to mention that they are not known for $n \geq 10$ and n not a perfect square number of general points in $\mathbb{P}^2$. The famous conjecture of Nagata, see [25], predicts that if $\mathcal{I}$ is the saturated ideal of a set of $n \geq 10$ general points in $\mathbb{P}^2$, then $\widehat{\alpha}(\mathcal{I}) = \sqrt{n}$. The Conjecture holds if n is a square of an integer. A lot of effort has been put in proving the Conjecture in any other case but all these attempts failed so far. The best uniform result available for an arbitrary number $n \geqslant 10$ is that

$$\lfloor \sqrt{n} \rfloor \leq \widehat{\alpha}(\mathcal{I}) \leq \sqrt{n}.$$

As a preparation for what we need in the next chapter let us now define star configurations, see [7].

Definition 3.4 (***Star configuration***)
Let N, r and s be positive integers with $1 \leq r \leq min\{N, s\}$. Let $F = \{F_1, \ldots, F_s\}$ be a set of homogeneous forms in $R = \mathbb{K}[x_0, \ldots, x_N]$ such that all subsets of F of cardinality less than or equal to $r + 1$ are regular sequences in R. Define an ideal of R by setting

$$\mathcal{I}_{r,F} = \bigcap_{1 \leq i_1 < \cdots < i_r \leq s} (F_{i_1}, \ldots, F_{i_r}).$$

The vanishing locus $V(\mathcal{I}_{r,F})$ in $\mathbb{P}^N$ is called a *star configuration* of type (r, s) of codimension r complete intersection subvarieties in $\mathbb{P}^N$ determined by F.

When the forms $F_1, \ldots, F_s$ are all linear, we will write $L = \{L_1, \ldots, L_s\}$ instead of $F = \{F_1, \ldots, F_s\}$, and we call the vanishing locus $\mathbb{X}$ of $\mathcal{I}_{r,L}$ a *linear star configuration* in $\mathbb{P}^N$ of type (r, s).

Remark 3.5
For any set of lines $L = \{\ell_1, \ldots, \ell_s\}$ in $\mathbb{P}^2$, with no three lines meeting in a point, the set $V(\mathcal{I}_{2,L})$ is a linear star configuration of points.

In particular a linear plane star configuration has a binomial $\binom{s}{2}$ number of points.

Example 3.6
Consider 3 non-collinear points from Example 3.3. This configuration is a star configuration in $\mathbb{P}^2$ of type $(2, 3)$ with $L_1 : \{x = 0\}$, $L_2 : \{y = 0\}$ and $L_3 : \{z = 0\}$. We have $V(I_2, \{\{x = 0\}, \{y = 0\}, \{z = 0\}\})$ in $\mathbb{P}^2$. Every point from the set of $\binom{3}{2} = 3$ points is an intersection of a pair of these three lines.

3.2 Resurgence

For a non-trivial homogeneous ideal $\mathcal{I}$ in the ring R of polynomials there is $\mathcal{I}^r \subset \mathcal{I}^{(m)}$ if and only if $r \geq m$. The celebrated result of Ein, Lazarsfeld and Smith [14, Theorem A] asserts that there is also some uniformity in the reverse containment.

Theorem 3.7 (*Ein, Lazarsfeld, Smith*)
Let $\mathcal{I} \subset \mathbb{C}[x_0, \ldots, x_N]$ be a homogeneous ideal. Then the containment

$$I^{(m)} \subset \mathcal{I}^r$$

is guaranteed for all $m \geq Nr$.

The bound provided in the Theorem is not sharp in general and it is an intriguing question to investigate how far m and r can go apart. To address this question Harbourne and Bocci in [5] introduced a new invariant measuring in effect discrepancy between ordinary and symbolic powers of ideals.

Definition 3.8 (*Resurgence*)
Let $\mathcal{I}$ be a homogeneous ideal in the ring of polynomials. The *resurgence of* $\mathcal{I}$ is the real number

$$\rho(\mathcal{I}) := \sup\left\{\frac{m}{r} : \mathcal{I}^{(m)} \nsubseteq \mathcal{I}^r\right\}.$$

As usual there is an asymptotic cousin of the above invariant defined.

Definition 3.9 (***Asymptotic resurgence***)

Let $\mathcal{I}$ be a homogeneous ideal in the ring of polynomials. The *asymptotic resurgence* of $\mathcal{I}$ is the real number

$$\widehat{\rho}(\mathcal{I}) := \sup\left\{\frac{m}{r} : \mathcal{I}^{(ms)} \nsubseteq \mathcal{I}^{rs} for s \gg 0\right\}.$$

Since we are principally interested in fat point schemes, we summarize main properties of both invariants under this assumption.

Theorem 3.10

Let $Z = m_1P_1 + \ldots + m_sP_s$ be a fat point scheme in $\mathbb{P}^N$ and let $\mathcal{I} = \mathcal{I}(Z)$. Then we have:

a) $1 \leqslant \rho(\mathcal{I}) \leqslant N$;

b) if $\frac{m}{r} < \frac{\alpha(\mathcal{I})}{\widehat{\alpha}(\mathcal{I})}$, *then for* $t \gg 0$ *there is* $\mathcal{I}^{(mt)} \nsubseteq \mathcal{I}^{(rt)}$;

c) if $\frac{m}{r} \geq \frac{\operatorname{reg}(\mathcal{I})}{\widehat{\alpha}(\mathcal{I})}$, *then* $\mathcal{I}^{(m)} \subseteq \mathcal{I}^r$;

d)

$$\frac{\alpha(\mathcal{I})}{\widehat{\alpha}(\mathcal{I})} \leqslant \rho(\mathcal{I}) \leqslant \frac{\operatorname{reg}(\mathcal{I})}{\widehat{\alpha}(\mathcal{I})},$$

and

$$\frac{\alpha(\mathcal{I})}{\widehat{\alpha}(\mathcal{I})} = \rho(\mathcal{I}) \quad \text{if } \alpha(\mathcal{I}) = \operatorname{reg}(\mathcal{I}).$$

Proof. See Theorem 3.2.4 [22]. □

Example 3.11

We continue Example 3.3 of 3 non-collinear points in $\mathbb{P}^2$:

$$P_1 = [1:0:0], \qquad P_2 = [0:1:0], \qquad P_3 = [0:0:1].$$

We have $\mathcal{I} = \mathcal{I}(P_1) \cap \mathcal{I}(P_2) \cap \mathcal{I}(P_3) = \langle xy, xz, yz\rangle$. By Example 1.25, we have that $reg(\mathcal{I}) = 2$.

We want to show first that $\widehat{\alpha}(\mathcal{I}) \leqslant \frac{3}{2}$. The divisor $L_1 \cup L_2 \cup L_3$ with $L_1 : \{x = 0\}$, $L_2 : \{y = 0\}$ and $L_3 : \{z = 0\}$ vanishes at every point of the configuration with multiplicity 2. By Theorem 1.13 $xyz \in \mathcal{I}^{(2)}$ and we get the assertion.

Turning to the lower bound we will illustrate in this example our general strategy which builds upon Bezout's Theorem. To this end assume that there exists a positive integer m and there exists a curve $\mathcal{C}$ of degree $\lceil\frac{3}{2}m\rceil - 1$ vanishing at every of the 3 points of the configuration with multiplicity at least m. By Theorem 1.30 either L_1 is a component of C or

$$\left\lceil \frac{3}{2}m \right\rceil - 1 = C.L_1 \geq m + m = 2m,$$

which is obviously excluded.

The same argument applies to the other two lines L_2 and L_3. Hence the union of all the three lines is contained in C. Thus we can subtract the divisor $L = L_1 + L_2 + L_3$ out of C and obtain a new curve C_1 with $\deg(C_1) = \left\lceil \frac{3}{2}m \right\rceil - 1 - 3$ and multiplicity of C in each point at least $m - 2$.

Repeating the same argument $k \geq 2$ times we obtain a curve C_k of degree $\left\lceil \frac{3}{2}m \right\rceil - 1 - 3k \geq 1$ and multiplicity at least $m - 2k$ in each of points in this configuration. Applying Theorem 1.30 we get that either

$$\left\lceil \frac{3}{2}m \right\rceil - 1 - 3k \geqslant 2(m - 2k)$$

which is not possible because $m \geq 2k$ or L is a component of C_k.

Since we cannot subtract L from C forever, we get a contradiction with the assumption that C exists.

Hence, there is no element in $\mathcal{I}^{(m)}$ of degree lower than $\left\lceil \frac{3}{2}m \right\rceil$. Consequently, the Waldschmidt constant for this configuration of points is equal to $\frac{3}{2}$.

As a consequence, by Theorem 3.10 we have

$$\frac{2}{\frac{3}{2}} \leqslant \rho(\mathcal{I}) \leqslant \frac{2}{\frac{3}{2}},$$

so the resurgence $\rho(\mathcal{I}) = \frac{4}{3}$.

Next we consider a set of points in $\mathbb{P}^2$ such that Theorem 3.10 does not provide the exact value of the resurgence of its ideal.

Nonexample 3.12 (*Dual Hesse configuration*)
Consider the following 12 points in $\mathbb{P}^2$:

$$\begin{array}{lll}
P_1 = [1:0:0], & P_2 = [0:1:0], & P_3 = [0:0:1], \\
P_4 = [1:1:1], & P_5 = \left[1:\varepsilon:\varepsilon^2\right], & P_6 = \left[1:\varepsilon^2:\varepsilon\right], \\
P_7 = [\varepsilon:1:1], & P_8 = [1:\varepsilon:1], & P_9 = [1:1:\varepsilon], \\
P_{10} = \left[\varepsilon^2:1:1\right], & P_{11} = \left[1:\varepsilon^2:1\right], & P_{12} = \left[1:1:\varepsilon^2\right].
\end{array}$$

where ε is a primitive root of 1 of order 3. These points are all intersection points of the following 9 lines:

$$L_1 : x - y = 0, \qquad L_2 : y - z = 0, \qquad L_3 : z - x = 0,$$

$$L_4 : x - \varepsilon y = 0, \qquad L_5 : y - \varepsilon z = 0, \qquad L_6 : z - \varepsilon x = 0,$$
$$L_7 : x - \varepsilon^2 y = 0, \qquad L_8 : y - \varepsilon^2 z = 0, \qquad L_9 : z - \varepsilon^2 x = 0.$$

Denote by $\mathcal{I}$ ideal of those points, namely $\mathcal{I} = \mathcal{I}(P_1) \cap \mathcal{I}(P_2) \cap \cdots \cap \mathcal{I}(P_{12})$. We have that $\alpha(\mathcal{I}) = 2$, $\widehat{\alpha}(\mathcal{I}) = \frac{3}{2}$, $\rho(\mathcal{I}) = \frac{3}{2}$ and $\operatorname{reg}(\mathcal{I}) = 5$ (see proof in [13], Theorem 2.1). However applying Theorem 3.10 we get

$$\frac{2}{\frac{3}{2}} \leqslant \rho(\mathcal{I}) \leqslant \frac{5}{\frac{3}{2}},$$

so that neither the lower bound, nor the upper bound are sharp in this case. Moreover, the upper bound is in fact even worse than the general upper bound valid for arbitrary ideals $\mathcal{J}$ of points in $\mathbb{P}^2$, namely $\rho(\mathcal{J}) \leq 2$.

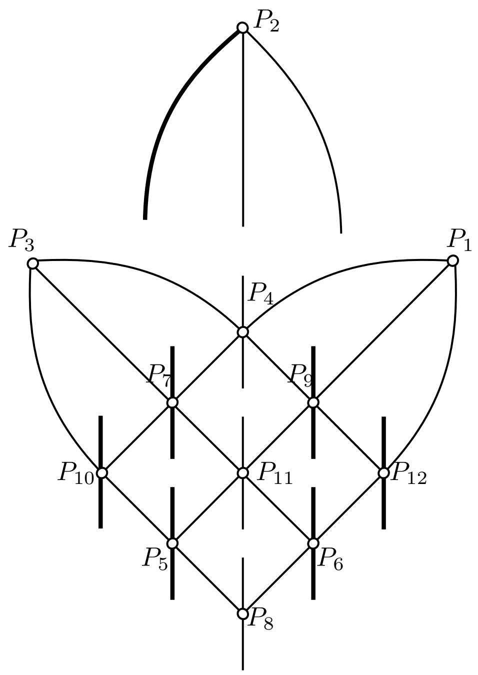


Figure 3.2: The dual Hesse configuration

4 Waldschmidt constants of symmetric sets of points in projective spaces

Our motivation to study Waldschmidt constants comes from the article from 2018: "Negative Curves on Symmetric Blowups of the Projective Plane, Resurgences, and Waldschmidt Constants". The authors focus on the Klein configuration which consists of 21 lines in $\mathbb{P}^2$ which meet in 21 quadruple points and 28 triple points, so that there are 49 points altogether. In this article authors consider also the Wiman configuration with 201 points and 45 lines which meet in 36 quintuple points, 45 quadruple points, and 120 triple points.

They tried to compute the Waldschmidt constant and the asymptotic resurgence for the mentioned configurations and presented the following theorem [3] (Theorem 1.1.):

Theorem 4.1

For the Klein configuration $\mathcal{K}$ of 21 lines, we have

$$6.480 \approx \frac{661}{102} \leqslant \widehat{\alpha}(\mathcal{I}_{\mathcal{K}}) \leqslant 6.5.$$

For the Wiman configuration $\mathcal{W}$ of 45 lines, we have

$$\widehat{\alpha}(\mathcal{I}_{\mathcal{W}}) = \frac{27}{2}.$$

To prove this theorem the authors emphasize the importance of the invariant theory. The proof of the upper estimate for the Waldschmidt constant for the above configurations is based on constructing curves in symbolic powers of $\mathcal{I}$. In the first step the authors define a divisor class D_k and specify the dimension of the linear system$|D_k|$. Typically D_k is created by taking appropriate union of configuration lines. This works in both cases of the Klein configuration and the Wiman configuration and provides an upper bound.

Getting the lower bound is much more demanding. Their approach is to prove that certain G-invariant divisor classes on the blowup X are nef. This prohibits the existence of curves in $\mathcal{I}^{(m)}$ with too low degree, as they would intersect the nef divisor negatively.

In the next section we compute the exact value of the Waldschmidt constant of the H_3 root system in $\mathbb{P}^2$. However we show that H_3 arises as a star configuration and the symmetries are irrelevant for the invariant we are interested in. Nevertheless, this configuration is highly symmetric as Klein configuration and Wiman configuration of lines in projective plane are.

Then we focus on D_4, B_4, F_4 and H_4 configurations in projective space $\mathbb{P}^3$. This is an innovative part of our work, as Waldschmidt constants of point configurations in higher dimensional projective spaces have not been studied so far. The configurations of points we study here are derived from root system. In this chapter we provide values of Waldschmidt constants of the above-mentioned configurations and we justify them with a theoretical proof. Only for the H_4 configuration we provide a conjectural value based on computer experiments.

All calculations carried out here were motivated and checked with the computer algebra system Singular [11] (compare the code in the Appendix).

We conclude this small introduction recalling an fundamental in the projective geometry.

Definition 4.2 (***Cross-ratio***)
Let $P_1, \ldots, P_4$ be mutually distinct points on the projective line $\mathbb{P}^1$. There exist numbers p, q, r, s such that

$$P_3 = p \cdot P_1 + q \cdot P_2, \quad P_4 = r \cdot P_1 + s \cdot P_2.$$

In this parameterization of the line we have

$$P_1 = (1:0), \quad P_2 = (0:1), \quad P_3 = (p:q), \quad P_4 = (r:s).$$

Then the cross-ratio of so ordered points is the number

$$\mathrm{DV}(P_1, P_2; P_3, P_4) = \frac{det(P_1P_3)}{det(P_1P_4)} : \frac{det(P_2P_4)}{det(P_2P_3)}.$$

4.1 H_3 root system

In this part, we describe the H_3 configuration. This configuration comes from a root system.

Definition 4.3

The set $Z(H_3)$ is a set of 15 points, which can be assigned the following coordinates:

$$\begin{array}{lll} P_1 = [1:0:0]\,, & P_2 = [0:1:0]\,, & P_3 = [0:0:1]\,, \\ P_4 = \left[1:\varphi:\varphi^2\right], & P_5 = \left[-1:\varphi:\varphi^2\right], & P_6 = \left[1:-\varphi:\varphi^2\right], \\ P_7 = \left[1:\varphi:-\varphi^2\right], & P_8 = \left[\varphi:-\varphi^2:1\right], & P_9 = \left[-\varphi:\varphi^2:1\right], \\ P_{10} = \left[\varphi:\varphi^2:-1\right], & P_{11} = \left[\varphi:\varphi^2:1\right], & P_{12} = \left[\varphi^2:1:-\varphi\right], \\ P_{13} = \left[\varphi^2:-1:\varphi\right], & P_{14} = \left[-\varphi^2:1:\varphi\right], & P_{15} = \left[\varphi^2:1:\varphi\right]. \end{array}$$

where we have $\varphi^2-\varphi-1 = 0$, so that φ is the golden ratio. The associated configuration together with 6 lines cutting it out is visualised at Figure 4.1. The 6 lines which pass through 5 configuration points each have the following equations:

$$\begin{array}{lll} L_1 : y-(\varphi-1)z = 0, & L_2 : y+(\varphi-1)z = 0, & L_3 : x-\varphi z = 0, \\ L_4 : x+\varphi z = 0, & L_5 : x-(\varphi-1)y = 0, & L_6 : x+(\varphi-1)y = 0. \end{array}$$

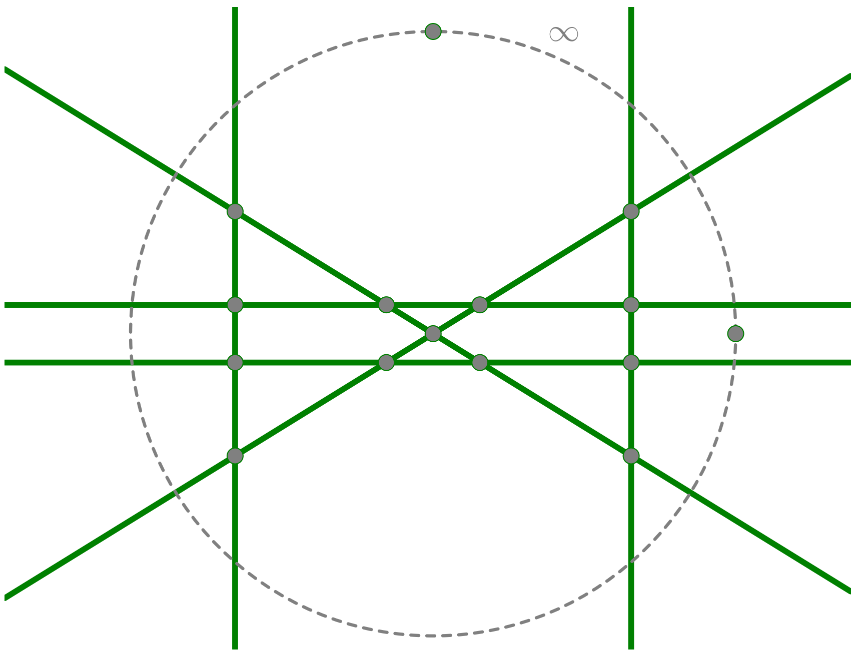


Figure 4.1: Line configuration associated to $Z(H_3)$

Theorem 4.4

The Waldschmidt constant of the H_3 configuration of points is equal to 3.

Remark 4.5

The H_3 configuration is an instance of a star configuration (see Definition 3.4) determined by 6 lines. Hence the result of Theorem 4.4 follows from general results in Subsection 4.2 in [12] or Proposition 2.9 in [20].

For the sake of completeness we provide an independent proof below.

PROOF. Consider the divisor $L_1 + \cdots + L_6$. It vanishes at every point of the configuration with multiplicity 2, so that using Theorem 1.13 we get

$$\widehat{\alpha}(\mathcal{I}) \leq \frac{6}{2} = 3.$$

Turning to a lower bound we follow our usual strategy. Let us assume that for some positive integer m there exists a curve $\mathcal{C}$ of degree $3m-1$ vanishing at every of the 15 configuration points with multiplicity at least m. Let us also assume that m is the lowest integer with the property that

$$\frac{\alpha(\mathcal{I}^{(m)})}{m} < 3.$$

By Theorem 1.30 L_1 is a component of C because the alternative case would imply

$$3m - 1 = C.L_1 \geq 5m,$$

which is clearly not possible.

The same argument applies to any other line $L_2, \ldots, L_6$. Hence their union is contained in C. Thus we can take the divisor $L = L_1 + \ldots + L_6$ out of C and obtain a new curve C_1 with $\deg(C_1) = 3m - 1 - 6 = 3(m-2) - 1$ and multiplicity of C in each point of H_3 at least $m-2$. But the existence of C_1 implies

$$\frac{\alpha(\mathcal{I}^{(m-2)})}{m-2} < 3$$

contradicting the minimality of m stipulated above. $\square$

In the next sections we consider Waldschmidt constants of some configurations of points in $\mathbb{P}^3$. We begin with the simplest root system in $\mathbb{P}^3$.

4.2 D_4 root system

In this section, we compute the Waldschmidt constant of the D_4 root system. Before we start let us make a remark on certain configuration of points in $\mathbb{P}^2$.

Remark 4.6

Consider the following 6 points in $\mathbb{P}^2$:

$$P_1 = [1:-1:0], \quad P_2 = [1:0:1], \quad P_3 = [1:0:-1],$$
$$P_4 = [0:1:1], \quad P_5 = [0:1:-1], \quad P_6 = [1:1:0]$$

and let $\mathcal{I}$ be their saturated ideal. We have that $\widehat{\alpha}(\mathcal{I}) = 2$. This is another instance of a star configuration determined by the following 4 lines:

$$L_1 : x - y - z = 0, \qquad L_2 : x + y - z = 0,$$
$$L_3 : x - y + z = 0, \qquad L_4 : x + y + z = 0.$$

This configuration is visualised in Figure 4.2.

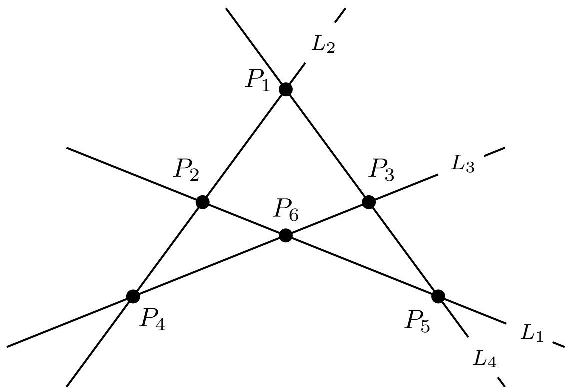


Figure 4.2: Configuration of 6 points

Up to a projective transformation the points in the D_4 configuration may be assumed to have the following coordinates:

$$P_1 = [1 : -1 : 0 : 0], \qquad P_2 = [0 : 1 : -1 : 0], \qquad P_3 = [0 : 0 : 1 : -1],$$
$$P_4 = [0 : 0 : 1 : 1], \qquad P_5 = [1 : 0 : -1 : 0], \qquad P_6 = [0 : 1 : 0 : -1],$$
$$P_7 = [0 : 1 : 0 : 1], \qquad P_8 = [1 : 0 : 0 : -1], \qquad P_9 = [1 : 0 : 0 : 1],$$
$$P_{10} = [0 : 1 : 1 : 0], \qquad P_{11} = [1 : 0 : 1 : 0], \qquad P_{12} = [1 : 1 : 0 : 0].$$

Theorem 4.7
The Waldschmidt constant of the D_4 configuration of points is 2.

PROOF.
We will construct a divisor actually computing the value 2. To this end let

$$H_1 := \{x = 0\}, \quad H_2 := \{y = 0\}, \quad H_3 := \{z = 0\}, \quad H_4 := \{w = 0\}.$$

For $H = H_1 + H_2 + H_3 + H_4$ we have $\deg(H) = 4$ and multiplicity of H in each point of D_4 is exactly 2 (see table 4.2). This implies that

$$\widehat{\alpha}(\mathcal{I}) \leq 2.$$

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}
H_1		+	+	+		+	+			+		
H_2			+	+	+			+	+		+	
H_3	+					+	+	+	+			+
H_4	+	+			+					+	+	+

Table 4.1: Incidences of points and planes

Suppose now that for some positive integer m there exists a surface S of degree lower than $2m$ vanishing at all 12 points of the configuration with multiplicity at least m. By the symmetry there are two options: either S contains each of the planes H_1, H_2, H_3, H_4 or none of them.

In the later case, for example for the plane H_4 to fix the notation, the intersection of S and H_4 is a curve of degree lower than $2m$ which vanishes in the 6 points from D_4 contained in this plane to the multiplicity at least m. It is not possible, because the Waldschmidt constant for these 6 points is equal to 2 by Remark 4.6. The same argument works for planes H_1, H_2 and H_3.

Thus S contains all planes H_1, H_2, H_3, H_4. This implies that $S-(H_1+H_2+H_3+H_4)$ has degree lower than $2(m-2)$ and vanishes at all 12 points of the configuration with multiplicity at least $m-2$. But these numbers allow us to conclude that H must be again a component of $S-H$. Since this goes forever, we get a contradiction with the assumption that S exists.

Consequently, the Waldschmidt constant for the D_4 configuration of 12 points is equal to 2. $\square$

4.3 B_4 root system

The next configuration extends the D_4 configuration by adding four more points.

Definition 4.8

The set $Z(B_4)$ consists of the following 16 points:

$$P_1 = [1:-1:0:0], \qquad P_2 = [0:1:-1:0], \qquad P_3 = [0:0:1:-1],$$
$$P_4 = [0:0:1:1], \qquad P_5 = [1:0:-1:0], \qquad P_6 = [0:1:0:-1],$$
$$P_7 = [0:1:0:1], \qquad P_8 = [1:0:0:-1], \qquad P_9 = [1:0:0:1],$$

$$P_{10} = [0 : 1 : 1 : 0]\,, \qquad P_{11} = [1 : 0 : 1 : 0]\,, \qquad P_{12} = [1 : 1 : 0 : 0]\,,$$
$$P_{13} = [1 : 1 : 1 : 1]\,, \qquad P_{14} = [-1 : -1 : 1 : 1]\,, \qquad P_{15} = [-1 : 1 : 1 : -1]\,,$$
$$P_{16} = [-1 : 1 : -1 : 1]\,.$$

The configuration of these points is visualized in Figure 4.3.

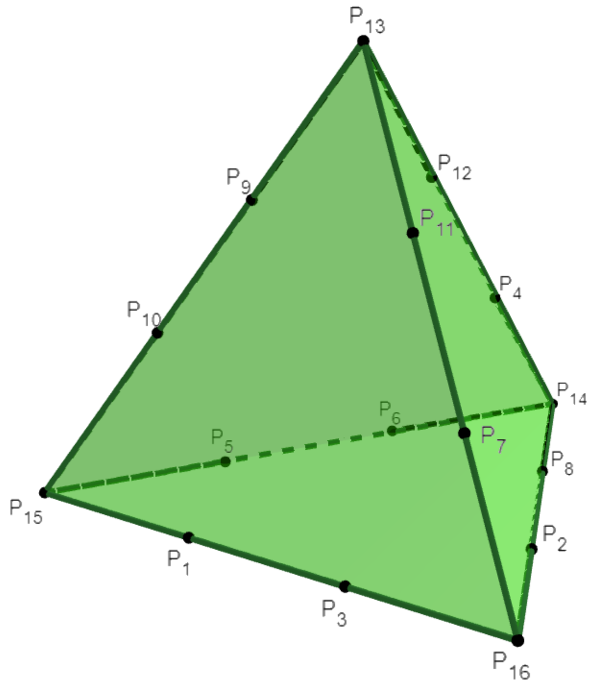


Figure 4.3: The B_4 configuration of points and the coordinate tetrahedron

The points in B_4 can be therefor considered as vertices of a tetrahedron together with two other points on each edge of this solid. Since there are altogether 4 points on each edge, it is natural to wonder about their cross-ration. By symmetry, it is the same on each edge, so we consider only a specific 4-tuple: $P_{13}, P_{12}, P_4, P_{14}$.

We have

$$[0 : 0 : 1 : 1] = p[1 : 1 : 1 : 1] + q[1 : 1 : 0 : 0]$$

with $p = 1, q = -1$ and

$$[-1 : -1 : 1 : 1] = r[1 : 1 : 1 : 1] + s[1 : 1 : 0 : 0]$$

with $r = 1$ and $s = -2$.

By Definition 4.2 the *cross-ratio* of this 4-tuple is

$$\frac{-1 \cdot 1}{1 \cdot (-2)} = \frac{1}{2}.$$

That in turn implies that the points are *harmonic*.

Theorem 4.9
The Waldschmidt constant for B_4 configuration of points is equal to 2.

PROOF. The configuration D_4 has 12 points and B_4 has 16 points. We can see that the $D_4 \subseteq B_4$. We added 4 points to the previous configuration, which already had the Waldschmidt constant 2. If we add points we expect the Waldschmidt constant to increase, so that in any case it must be

$$\widehat{\alpha}(\mathcal{I}_{B_4}) \geq 2.$$

We consider the following four planes:

$$\begin{aligned}
\Pi_1 &: x + y + z + w = 0, \\
\Pi_2 &: x - y + z - w = 0, \\
\Pi_3 &: x - y - z + w = 0, \\
\Pi_4 &: x + y - z - w = 0.
\end{aligned}$$

Each of the planes contains 9 configuration points. Each configuration point is contained in at least two of the planes and the points $P_{13}, P_{14}, P_{15}, P_{16}$ are contained in three of them.

Hence, the polynomial

$$P(x : y : z : w) = (x + y + z + w)(x - y + z - w)(x - y - z + w)(x + y - z - w)$$

belongs in any case to the second symbolic power of $\mathcal{I}(B_4)$ and $\deg(P) = 4$, so that

$$\widehat{\alpha}(\mathcal{I}_{B_4}) \leq 2.$$

Consequently, the Waldschmidt constant for the B_4 configuration of 16 points is equal to 2. □

4.4 The F_4 root system

In this section, we consider an even bigger root system F_4. It contains the B_4 root system and consequently also the D_4. Computing its Waldschmidt constant has turned to be much more complicated that for the two systems considered before.

Let us start with the definition of a *grid* [29].

Definition 4.10 (*Grid*)

Let a and b be positive integers. A set Z of $a \cdot b$ points in $\mathbb{P}^3$ is an $(a, b)-$*grid* if there exist two sets of lines $L_1, \dots, L_a$ and $M_1, \dots, M_b$ such that

- lines in each of the sets are pairwise skew;
- each pair of lines, one from one set and one from the other intersect in a point of Z.

Thus

$$Z = \{L_i \cap M_j,\ i = 1, \dots, a,\ j = 1, \dots, b\}.$$

An example of a $(2, 2)$-grid is visualised in Figure 4.4.

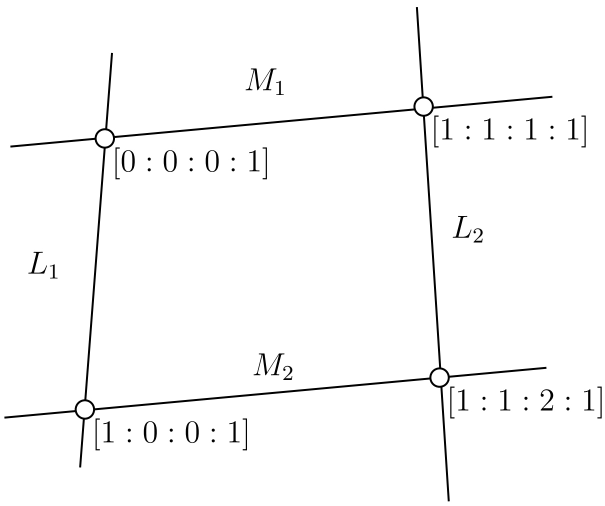


Figure 4.4: A $(2, 2)$ - grid

Remark 4.11

Note that any 4 points in $\mathbb{P}^3$ in linear general position form a $(2, 2)$-grid.
More generally, any set of $2b$ points distributed by b on two skew lines forms a $(2, b)$-grid.
These cases are somewhat special. In general, being a grid imposes strong constrains on the geometry of the underlying set of points.

Now we turn attention to the main hero of this section. We use explicit coordinates rather than theoretical description because in this way we can much easier check various incidences among points in F_4 and some lines and planes in $\mathbb{P}^3$.

Definition 4.12

The set $Z(F_4)$ consists of 24 points in $\mathbb{P}^3$, which can be assigned the following coordinates:

$$P_1 = [1 : -1 : 0 : 0], \qquad P_2 = [0 : 1 : -1 : 0], \qquad P_3 = [0 : 0 : 1 : -1],$$

$$
\begin{array}{lll}
P_4 = [0:0:1:1]\,, & P_5 = [1:0:-1:0]\,, & P_6 = [0:1:0:-1]\,, \\
P_7 = [0:1:0:1]\,, & P_8 = [1:0:0:-1]\,, & P_9 = [1:0:0:1]\,, \\
P_{10} = [0:1:1:0]\,, & P_{11} = [1:0:1:0]\,, & P_{12} = [1:1:0:0]\,, \\
P_{13} = [1:1:1:1]\,, & P_{14} = [-1:-1:1:1]\,, & P_{15} = [-1:1:1:-1]\,, \\
P_{16} = [-1:1:-1:1]\,, & P_{17} = [1:1:1:-1]\,, & P_{18} = [1:1:-1:1]\,, \\
P_{19} = [1:-1:1:1]\,, & P_{20} = [1:-1:-1:-1]\,, & P_{21} = [1:0:0:0]\,, \\
P_{22} = [0:1:0:0]\,, & P_{23} = [0:0:1:0]\,, & P_{24} = [0:0:0:1]\,.
\end{array}
$$

This set contains some $(4,4)$-grids. For example there are 16 points in F_4 contained in the smooth quadric Q defined by equation $xw - yz = 0$. This subset is a $(4,4)$-grid visualised in Figure 4.5. The remaining 8 points split in two 4-tuples of collinear points. This is also indicated in Figure 4.5 together with equations defining these two lines and lines building the grid.

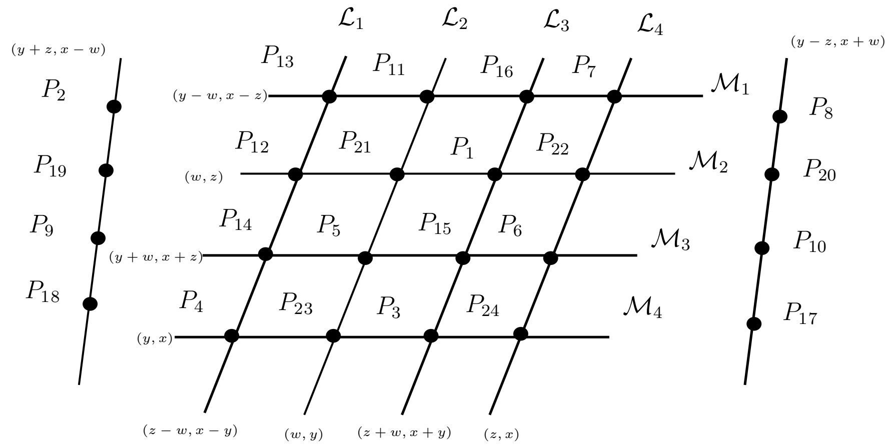


Figure 4.5: The F_4 configuration of 24 points

Taking coordinates of points P_i as coefficients of linear equations, we obtain the following set of 24 planes (note that the numbering does not much the aforementioned way to derive the equations).

Π_1: $x + y = 0$,	Π_2: $y + z = 0$,
Π_3: $z + w = 0$,	Π_4: $z - w = 0$,
Π_5: $x + z = 0$,	Π_6: $y + w = 0$,
Π_7: $y - w = 0$,	Π_8: $x + w = 0$,
Π_9: $x - w = 0$,	Π_{10}: $y - z = 0$,
Π_{11}: $x - z = 0$,	Π_{12}: $x - y = 0$,
Π_{13}: $-x - y + z + w = 0$,	Π_{14}: $-x + y + z - w = 0$,
Π_{15}: $x - y + z - w = 0$,	Π_{16}: $x + y + z + w = 0$,
Π_{17}: $x - y + z + w = 0$,	Π_{18}: $-x + y + z + w = 0$,
Π_{19}: $-x - y + z - w = 0$,	Π_{20}: $x + y + z - w = 0$,
Π_{21}: $x = 0$,	Π_{22}: $y = 0$,
Π_{23}: $z = 0$,	Π_{24}: $w = 0$.

Table 4.2: The 24 planes associated to the F_4 configuration

Every plane Π_i contains exactly 9 points from the F_4 configuration. This arrangement is self-dual in the sense that each of 24 points P_i lies on exactly 9 planes. In Table 4.3, we present a table of incidences between the above planes and the points.

Π_1:	P_1	P_3	P_4	P_{15}	P_{16}	P_{19}	P_{20}	P_{23}	P_{24}
Π_2:	P_2	P_8	P_9	P_{14}	P_{16}	P_{18}	P_{19}	P_{21}	P_{24}
Π_3:	P_1	P_3	P_{12}	P_{15}	P_{16}	P_{17}	P_{18}	P_{21}	P_{22}
Π_4:	P_1	P_4	P_{12}	P_{13}	P_{14}	P_{19}	P_{20}	P_{21}	P_{22}
Π_5:	P_5	P_6	P_7	P_{14}	P_{15}	P_{18}	P_{20}	P_{22}	P_{24}
Π_6:	P_5	P_6	P_{11}	P_{14}	P_{15}	P_{17}	P_{19}	P_{21}	P_{23}
Π_7:	P_5	P_7	P_{11}	P_{13}	P_{16}	P_{18}	P_{20}	P_{21}	P_{23}
Π_8:	P_2	P_8	P_{10}	P_{14}	P_{16}	P_{17}	P_{20}	P_{22}	P_{23}
Π_9:	P_2	P_9	P_{10}	P_{13}	P_{15}	P_{18}	P_{19}	P_{22}	P_{23}
Π_{10}:	P_8	P_9	P_{10}	P_{13}	P_{15}	P_{17}	P_{20}	P_{21}	P_{24}
Π_{11}:	P_6	P_7	P_{11}	P_{13}	P_{16}	P_{17}	P_{19}	P_{22}	P_{24}
Π_{12}:	P_3	P_4	P_{12}	P_{13}	P_{14}	P_{17}	P_{18}	P_{23}	P_{24}
Π_{13}:	P_1	P_3	P_7	P_9	P_{10}	P_{11}	P_{13}	P_{15}	P_{16}
Π_{14}:	P_2	P_4	P_7	P_8	P_{11}	P_{12}	P_{13}	P_{14}	P_{16}
Π_{15}:	P_4	P_5	P_6	P_9	P_{10}	P_{12}	P_{13}	P_{14}	P_{15}
Π_{16}:	P_1	P_2	P_3	P_5	P_6	P_8	P_{14}	P_{15}	P_{16}
Π_{17}:	P_3	P_5	P_7	P_8	P_{10}	P_{12}	P_{17}	P_{18}	P_{20}

Π_{18}:	P_2	P_3	P_6	P_9	P_{11}	P_{12}	P_{17}	P_{18}	P_{19}
Π_{19}:	P_1	P_4	P_6	P_8	P_{10}	P_{11}	P_{17}	P_{19}	P_{20}
Π_{20}:	P_1	P_2	P_4	P_5	P_7	P_9	P_{18}	P_{19}	P_{20}
Π_{21}:	P_2	P_3	P_4	P_6	P_7	P_{10}	P_{22}	P_{23}	P_{24}
Π_{22}:	P_3	P_4	P_5	P_8	P_9	P_{11}	P_{21}	P_{23}	P_{24}
Π_{23}:	P_1	P_6	P_7	P_8	P_9	P_{12}	P_{21}	P_{22}	P_{24}
Π_{24}:	P_1	P_2	P_5	P_{10}	P_{11}	P_{12}	P_{21}	P_{22}	P_{23}

Table 4.3: Planes and points incidences

Let us now consider one of the configuration planes for F_4, namely we take the plane Π_{24} defined by the equation $w = 0$. This plane contains, as already mentioned 9 points from F_4. We denote this set of points by Z_9:

$$P_1 = [1 : -1 : 0 : 0]\,, \qquad P_2 = [0 : 1 : -1 : 0]\,, \qquad P_5 = [1 : 0 : -1 : 0]\,,$$
$$P_{10} = [0 : 1 : 1 : 0]\,, \qquad P_{11} = [1 : 0 : 1 : 0]\,, \qquad P_{12} = [1 : 1 : 0 : 0]\,,$$
$$P_{21} = [1 : 0 : 0 : 0]\,, \qquad P_{22} = [0 : 1 : 0 : 0]\,, \qquad P_{23} = [0 : 0 : 1 : 0]\,.$$

In Figure 4.6 the set Z_9 is indicated in the plane Π_{24}, where the line $z = 0$ is taken as the line at infinity.

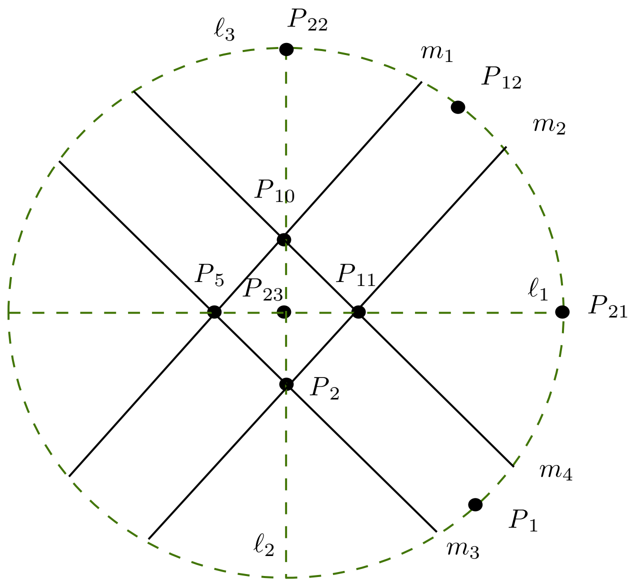


Figure 4.6: Visualization of Z_9 in Π_{24}

The points in Z_9 form a union of two star configurations, one defined by 3 lines ℓ_1, ℓ_2, ℓ_3. Each of them contains 4 points from the set Z_9. Their equations in the plane $w = 0$ are very nice:

$$\ell_1 : y = 0, \qquad \ell_2 : x = 0, \qquad \ell_3 : z = 0.$$

The other star configuration is defined by 4 lines $m_1, \dots, m_4$, which pass through 3 points each. They have the following equations:

$$m_1 : x - y + z = 0, \qquad m_2 : x - y - z = 0,$$
$$m_3 : x + y + z = 0, \qquad m_4 : x + y - z = 0.$$

This configuration is visualised in Figure 4.6.

The Waldschmidt constant for the first star configuration is equal to $\frac{3}{2}$. On the other hand, for the six points of the second star configuration, the Waldschmidt constant is equal to 2.

These configurations complement each other as follows: Six points coming from the second star configuration are distributed by two on the lines ℓ_1, ℓ_2, ℓ_3 forming the first star.

Theorem 4.13
The Waldschmidt constant for Z_9 configuration is equal to $\frac{5}{2}$.

PROOF. To begin with note that there are the following incidences:

$$P_5, P_{11} \in \ell_1, \quad P_2, P_{10} \in \ell_2, \quad P_1, P_{12} \in \ell_3.$$

Let $D_1 = \ell_1 + \ell_2 + \ell_3$ and $D_2 = m_1 + m_2 + m_3 + m_4$. We consider the order of vanishing of D_1 and D_2 in the points of Z_9.

point	P_1	P_2	P_5	P_{10}	P_{11}	P_{12}	P_{21}	P_{22}	P_{23}
order of vanishing for D_1	1	1	1	1	1	1	2	2	2
order of vanishing for D_2	2	2	2	2	2	2	0	0	0

Taking $\Delta = 2D_1 + D_2$ we obtain a curve of degree 10 vanishing at all 9 points to order 4. This implies:

$$\widehat{\alpha}(Z) \leq \frac{10}{4} = \frac{5}{2}.$$

Suppose that there is a curve γ of degree strictly less than $\frac{5}{2}m$ vanishing to order m at all points of Z_9. Without any loss of generality we may assume additionally $m \geq 5$.

Theorem 1.30 implies that either

$$\deg \gamma \deg \ell_i \geq \sum_{P \in Z} (\operatorname{mult}_P \gamma \cdot \operatorname{mult}_P \ell_i)$$

or ℓ_i is a component of γ.

In the first case we would have

$$\frac{5}{2}m > 4m,$$

which is not possible. So $\ell_i \subseteq \gamma$ for all $i = 1, 2, 3$.

Let $\gamma' = \gamma - D_1$. Then $\deg \gamma' < \frac{5m}{2} - 3$ and

$$\operatorname{mult}_P \gamma' \geq \begin{cases} m - 1 \text{ for } P \in \{P_1, P_2, P_5, P_{10}, P_{11}, P_{12}\} \\ m - 2 \text{ for } P \in \{P_{21}, P_{22}, P_{23}\}. \end{cases}$$

Applying Theorem 1.30 once again for the curves γ' and ℓ_i we obtain:

$$\frac{5}{2}m - 3 > 2(m-2) + 2(m-1)$$

which reduces to $2 > m$, which is clearly not possible, so D_1 is again a component of γ'. Thus γ contains $2D_1$.

By the same token we have that either

$$\deg \gamma \deg m_i \geq \sum_{P \in Z} (\operatorname{mult}_P \gamma \cdot \operatorname{mult}_P m_i)$$

or m_i is a component of γ. The first case implies

$$\frac{5}{2}m \geq 3m,$$

which is not possible. Consequently, $m_i \subseteq \gamma$ for all $i = 1, 2, 3, 4$, so that γ contains D_2.

Let $\gamma'' = \gamma - 2D_1 - D_2$. Then $\deg \gamma'' < \frac{5}{2}m - 10 = \frac{5}{2}(m-4)$ and $\operatorname{mult}_P \gamma'' \geq m - 4$ for all $P \in Z_9$.

Hence we are in the position to repeat the above considerations with m replaced with $m-4$. Since γ can contain only finitely many copies of Δ we arrive to a contradiction.

Consequently, we get that the Waldschmidt constant for the configuration Z_9 of 9 points is equal to $\frac{5}{2}$. $\square$

Example 4.14

We consider the minimal free resolution of the ideal $\mathcal{I} = \mathcal{I}(Z_9)$ of the above 9 points and obtain:

$$0 \to R^3(-6) \xrightarrow{\alpha^T} R(-4) \oplus R^6(-5) \xrightarrow{\beta^T} R(-1) \oplus R(-3) \oplus R^3(-4) \xrightarrow{\gamma} \mathcal{I} \to 0,$$

where

$$\alpha = \begin{bmatrix} y^2-z^2 & -x & w & 0 & 0 & 0 & 0 \\ x^2-z^2 & 0 & 0 & -y & w & 0 & 0 \\ x^2-y^2 & 0 & 0 & 0 & 0 & -z & w \end{bmatrix},$$

$$\beta = \begin{bmatrix} -xyz & w & 0 & 0 & 0 \\ -y^3z+yz^3 & 0 & w & 0 & 0 \\ 0 & -y^2+z^2 & x & 0 & 0 \\ -x^3z+xz^3 & 0 & 0 & w & 0 \\ 0 & -x^2+z^2 & 0 & y & 0 \\ -x^3y+xy^3 & 0 & 0 & 0 & w \\ 0 & -x^2+y^2 & 0 & 0 & z \end{bmatrix},$$

and

$\gamma = \begin{bmatrix} w & xyz & y^3z-yz^3 & x^3z-xz^3 & x^3y-xy^3 \end{bmatrix}$. It follows that

$$\operatorname{reg}(\mathcal{I}) = \max\{4-0;\ 3-0;\ 1-0;\ 5-1;\ 1-1;\ 3-1\} = 4.$$

Revoking Theorem 3.10 d) we conclude that the resurgence for the Z_9 configuration is

$$\rho(\mathcal{I}) = \frac{\alpha(\mathcal{I})}{\widehat{\alpha}(\mathcal{I})} = \frac{\operatorname{reg}(\mathcal{I})}{\widehat{\alpha}(\mathcal{I})} = \frac{8}{5}.$$

We come now to the main result of this part and of the whole thesis.

Theorem 4.15

The Waldschmidt constant of the F_4 configuration of points is equal to $\frac{8}{3}$.

PROOF. Keeping the notation from Table 4.2, we consider the divisor

$$D = \Pi_1 + \Pi_2 + \cdots + \Pi_{24} \tag{4.1}$$

of degree 24 vanishing at every point of the F_4 configuration with multiplicity 9. Then, we obtain

$$\widehat{\alpha}(\mathcal{I}) \leq \frac{24}{9} = \frac{8}{3}.$$

The rest of the proof amounts to justifying the inverse inequality

$$\widehat{\alpha}(\mathcal{I}) \geq \frac{8}{3}. \tag{4.2}$$

One could naively hope that the standard argument with the Bezout's Theorem applied to a divisor S violating (4.2) would suffice. By this we mean that taking the trace of S on one (hence on any) of the planes Π_i would violate the Waldschmidt constant of the configuration of points of F_4 contained in that plane, thus forcing the plane to be a component of S. However, since $\frac{8}{3} > \frac{5}{2}$ this approach cannot work.

As a remedy to this deficiency we need to consider additional collinearities among the configuration points. So, interestingly it is not enough to consider divisors only but we must (at least in our approach) pay attention to lower dimensional subvarieties as well.

To begin with, we consider lines passing through 3 points from the F_4 configuration. There are 32 such lines $t_1, \ldots, t_{32}$. We present ideals of these lines below:

$$\begin{aligned}
t_1 &= (w, x+y+z), & t_2 &= (z, x+y+w), & t_3 &= (z, x+y-w),\\
t_4 &= (w, x+y-z), & t_5 &= (x+z+w, x), & t_6 &= (y+z-w, x),\\
t_7 &= (w, x-y-z), & t_8 &= (y, x+z+w), & t_9 &= (y-z-w, x),\\
t_{10} &= (y, x-z-w), & t_{11} &= (y, x+z-w), & t_{12} &= (y-z+w, x),\\
t_{13} &= (y, x-z+w), & t_{14} &= (w, x-y+z), & t_{15} &= (z, x-y-w),\\
t_{16} &= (z, x-y+w), & t_{17} &= (y-z, x-z), & t_{18} &= (y-w, x-w),\\
t_{19} &= (z-w, x-w), & t_{20} &= (z-w, y-w), & t_{21} &= (y+w, x+w),\\
t_{22} &= (y+z, x+z), & t_{23} &= (z-w, y+w), & t_{24} &= (z-w, x+w),\\
t_{25} &= (z+w, y+w), & t_{26} &= (z+w, x-w), & t_{27} &= (y+w, x-w),\\
t_{28} &= (y-z, x+z), & t_{29} &= (z+w, x+w), & t_{30} &= (z+w, y-w),\\
t_{31} &= (y+z, x-z), & t_{32} &= (y-w, x+w). & &
\end{aligned}$$

In principle, the claims above can be verified by hand computations. We used Singular to facilitate dull calculations.

It is also useful to keep record of which points from the configuration are contained on which lines. These incidences are recorded in the list below.

$$\begin{aligned}
t_1 &: P_1, P_2, P_5, & t_2 &: P_1, P_6, P_8, & t_3 &: P_1, P_7, P_9,\\
t_4 &: P_1, P_{10}, P_{11}, & t_5 &: P_2, P_3, P_6, & t_6 &: P_2, P_4, P_7,\\
t_7 &: P_2, P_{11}, P_{12}, & t_8 &: P_3, P_5, P_8, & t_9 &: P_3, P_7, P_{10},\\
t_{10} &: P_3, P_9, P_{11}, & t_{11} &: P_4, P_5, P_9, & t_{12} &: P_4, P_6, P_{10},\\
t_{13} &: P_4, P_8, P_{11}, & t_{14} &: P_5, P_{10}, P_{12}, & t_{15} &: P_6, P_9, P_{12},\\
t_{16} &: P_7, P_8, P_{12}, & t_{17} &: P_{13}, P_{17}, P_{24}, & t_{18} &: P_{13}, P_{18}, P_{23},\\
t_{19} &: P_{13}, P_{19}, P_{22}, & t_{20} &: P_{13}, P_{20}, P_{21}, & t_{21} &: P_{14}, P_{17}, P_{23},\\
t_{22} &: P_{14}, P_{18}, P_{23}, & t_{23} &: P_{14}, P_{19}, P_{21}, & t_{24} &: P_{14}, P_{20}, P_{22},
\end{aligned}$$

$$t_{25} : P_{15}, P_{17}, P_{21}, \qquad t_{26} : P_{15}, P_{18}, P_{22}, \qquad t_{27} : P_{15}, P_{19}, P_{23},$$
$$t_{28} : P_{15}, P_{20}, P_{24}, \qquad t_{29} : P_{16}, P_{17}, P_{22}, \qquad t_{30} : P_{16}, P_{18}, P_{21},$$
$$t_{31} : P_{16}, P_{19}, P_{24}, \qquad t_{32} : P_{16}, P_{20}, P_{23}.$$

Now, an important observation is that intersecting these 32 lines with the plane $w = 0$, which is Π_{24} we obtain 4 new points:

$$Z_1 = [1 : 1 : 1 : 0] , \quad Z_2 = [1 : 1 : -1 : 0] , \quad Z_3 = [-1 : 1 : 1 : 0] , \quad Z_4 = [1 : -1 : 1 : 0] .$$

Pairs of these points determine 6 new lines $k_1, \ldots, k_6$, whose ideals are presented below:

$$k_1 = (w, x - y), \qquad k_2 = (w, y - z), \qquad k_3 = (w, x - z),$$
$$k_4 = (w, x + z), \qquad k_5 = (w, y + z), \qquad k_6 = (w, x + y).$$

These new lines pass also through some of the F_4 configurations points contained in the plane $\{w = 0\}$. Precise incidences are recorded in the table below. This table introduces also a convenient notation for sets of distinguished points. We have

$$\mathbb{Z} = \{Z_1, Z_2, Z_3, Z_4\} , \quad \mathbb{Q} = \{P_1, P_2, P_5, P_{10}, P_{11}, P_{12}\} , \quad \mathbb{P} = \{P_{21}, P_{22}, P_{23}\} .$$

Note that the points in $\mathbb{Q}$ are intersection points of pairs of m_i lines, whereas the points in $\mathbb{P}$ are intersection points of ℓ_i lines. Thus both sets are star configurations.

	$\mathbb{Z}$				$\mathbb{Q}$						$\mathbb{P}$		
	Z_1	Z_2	Z_3	Z_4	P_1	P_2	P_5	P_{10}	P_{11}	P_{12}	P_{21}	P_{22}	P_{23}
k_1	+	+								+			+
k_2	+		+					+			+		
k_3	+			+					+			+	
k_4		+	+				+					+	
k_5		+		+		+					+		
k_6			+	+	+								+

Thus on each line k_i through two points from $\mathbb{Z}$ there is a point P_j where two of m_i lines meet and a point Q_j where two of ℓ_i lines meet.

Completing the sets of points, there are three sets of distinguished lines:

1. $\mathbb{M} : m_1, m_2, m_3, m_4$;
2. $\mathbb{L} : \ell_1, \ell_2, \ell_3$;
3. $\mathbb{K} : k_1, k_2, k_3, k_4, k_5, k_6$.

Let S be a surface of degree d not containing the $\{w = 0\}$ plane and vanishing at all points from a set $\mathbb{X}$ to order at least $[\mathbb{X}]$, where $\mathbb{X}$ is one of the sets $\mathbb{P}, \mathbb{Q}$ or $\mathbb{Z}$. Restricting S to that plane we obtain a curve Γ of degree d. This curve passes through various points distinguished in that plane, so that by the Bezout theorem some lines determined by them are forced to be components of Γ. Consequently these lines can be subtracted from Γ and we may examine the residual curve, which is simpler because it has lower degree and its multiplicities in the relevant points are also lower. Somewhat informally, we call this procedure "a reduction game" on Γ. The key point here is to use the symmetry of the studied configuration. By this token, as soon as one of lines m_i, ℓ_i or k_i is forced to be a component of Γ, in fact the whole set $\mathbb{M}, \mathbb{L}$ or $\mathbb{K}$ must be a component. We will explain it in more details in the **special case** studied before. For now, let us observe how subtracting any of sets $\mathbb{M}, \mathbb{L}$ or $\mathbb{K}$ from Γ affects its degree and multiplicities in sets $\mathbb{P}, \mathbb{Q}$ or $\mathbb{Z}$.

For $\mathbb{M}$, we have:	For $\mathbb{L}$, we have:	For $\mathbb{K}$, we have:
$d \to d - 4;$	$d \to d - 3;$	$d \to d - 6;$
$[\mathbb{P}] \to [\mathbb{P}];$	$[\mathbb{P}] \to [\mathbb{P}] - 2;$	$[\mathbb{P}] \to [\mathbb{P}] - 2;$
$[\mathbb{Q}] \to [\mathbb{Q}] - 2;$	$[\mathbb{Q}] \to [\mathbb{Q}] - 1;$	$[\mathbb{Q}] \to [\mathbb{Q}] - 1;$
$[\mathbb{Z}] \to [\mathbb{Z}].$	$[\mathbb{Z}] \to [\mathbb{Z}].$	$[\mathbb{Z}] \to [\mathbb{Z}] - 3.$

So that for $2\mathbb{K} + 3\mathbb{M}$, we have:

$d \to d - 24;$

$[\mathbb{P}] \to [\mathbb{P}] - 4;$

$[\mathbb{Q}] \to [\mathbb{Q}] - 8;$

$[\mathbb{Z}] \to [\mathbb{Z}] - 6.$

We illustrate our strategy first in a specific special case before passing to the general statement.

Special case

We want to show that there is no surface S of degree $2399 = 24 \cdot 100 - 1$ vanishing at every of 24 points of the F_4 configuration with multiplicity $\geq 900 = 9 \cdot 100$. Let us assume to the contrary that such a surface exists and that the trace of this surface on the plane $\Pi_{24} = \{w = 0\}$ is a curve Γ. (If S contains the plane, then by symmetry it must contain the whole divisor D defined in (4.1). Taking $S - D$, we get another surface S' satisfying

$$\frac{\deg(S')}{\operatorname{mult}_{F_4} S'} < \frac{8}{3}$$

and we argue with that surface.) We begin with a reduction with respect to the line m_1.

By the Bezout theorem, we have that the curve Γ either contains the line m_1 or

$$2399 = S.m_1 \geqslant 3 \cdot 900,$$

which is clearly not possible.

We want to find the multiplicity k with which the curve Γ must contain the line m_1. To this end we want to determine the smallest k such that the following inequality holds:

$$2399 - k \geqslant 3 \cdot (900 - k).$$

It follows that

$$k \geqslant \left\lceil \frac{301}{2} \right\rceil = 151.$$

Of course, by symmetry, this is in fact the least multiplicity, such that the union of lines $\mathbb{M}$ appears as a component of Γ.

In the next step we perform a reduction of $(\Gamma - 151\mathbb{M})$ with respect to $\mathbb{L}$. Looking at the intersection of this curve with one of the lines ℓ_i, we obtain similarly as above:

$$1795 - k \geqslant 2(900 - k) + 2(598 - k).$$

This implies

$$k \geqslant \left\lceil \frac{1201}{3} \right\rceil = 401.$$

Subtracting we obtain a curve $\Gamma' = \Gamma - 151\mathbb{M} - 401\mathbb{L}$ of degree 592. Since

$$592 + 5 = 98 + 197 + 2 \cdot 151 \tag{4.3}$$

and the multiplicity of Γ' in points of $\mathbb{P}$ is at least 98, in points of $[\mathbb{Q}]$ is at least 197 and in points of $[\mathbb{Z}]$ is at least 151 we can continue the reduction game with the union of lines $\mathbb{K}$. Looking for the least multiplicity k such that Γ' contains $\mathbb{K}$ we get:

$$592 - k \geqslant 98 - k + 197 - k + 302 - 2k,$$

which amounts to

$$k \geqslant \left\lceil \frac{5}{3} \right\rceil = 2.$$

Now, it turns out, that we in the position to continue our game with lines in $\mathbb{M}$ since

$$580 - k \geqslant 3(195 - k)$$

gives

$$k \geqslant \left\lceil \frac{5}{2} \right\rceil = 3.$$

After the last two steps we obtain

$$568 + 5 = 94 + 189 + 2 \cdot 145, \tag{4.4}$$

where 568 is degree of the curve $\Gamma'' = \Gamma' - (2\mathbb{K} + 3\mathbb{M})$ with multiplicity at least 94 in points in $[\mathbb{P}]$, 189 in points of $[\mathbb{Q}]$ and 145 points of $[\mathbb{Z}]$. This means that (4.4) is basically the same condition as (4.3) and we can repeat removing the divisor $2\mathbb{K} + 3\mathbb{M}$ over and over again, at least as long as the residual curve retains positive multiplicities in all relevant points. With this particular data we can do this 23 times. In the two final two steps we subtract $2\mathbb{K}$ and $\mathbb{M}$ and conclude with a clear contradiction. The record of our reduction game is summarized in Table 4.4.

d	$[\mathbb{P}]$	$[\mathbb{Q}]$	$[\mathbb{Z}]$	reduction
2399	900	900	151	$151 \cdot \mathbb{M}$
1795	900	598	151	$401 \cdot \mathbb{L}$
592	98	197	151	$2 \cdot \mathbb{K}$
580	94	195	145	$3 \cdot \mathbb{M}$
568	94	189	145	$23 \cdot (2\mathbb{K} + 3\mathbb{M})$
16	2	5	7	$2 \cdot \mathbb{K}$
4	-2	3	1	$1 \cdot \mathbb{M}$
0		2	1	

Table 4.4: The reduction game in the special case

The general case

We basically mimic our strategy from the special case. Since the Waldschmidt constant is the limit of the sequence $\alpha(\mathcal{I}^{(m)})/m$, it is computed by any subsequence of m's. We restrict our attention to powers of 10. Thus it is enough to show that there is no surface of degree $24 \cdot 10^a - 1$ which vanishes to order at least $9 \cdot 10^a$ at all points of F_4. The existence of such a surface would imply

$$\widehat{\alpha}(\mathcal{I}(F_4)) < \frac{24 \cdot 10^a}{9 \cdot 10^a} = \frac{8}{3}.$$

So let us assume to the contrary that such a surface S exists. We want to show that it contains all plains Π_i, with $i = 1, \ldots, 24$.

We observe first that S contains every line t_i with $i = 1, \dots, 32$ with multiplicity at least $1/2 \cdot 3 \cdot 10^a + 1$. Indeed, it is the least value of k such that Bezout's inequality

$$24 \cdot 10^a - 1 - k \geq 3 \cdot (9 \cdot 10^a - k) \tag{4.5}$$

is satisfied. Consequently, if S does not contain e.g. Π_{24}, then the multiplicity of the curve $\Gamma = S \cap \Pi_{24}$ at the Z_i points is at least that big. Thus we have the initial data for our reduction game.

d	$[\mathbb{P}]$	$[\mathbb{Q}]$	$[\mathbb{Z}]$
$24 \cdot 10^a - 1$	$9 \cdot 10^a$	$9 \cdot 10^a$	$1/2 \cdot 3 \cdot 10^a + 1$

Table 4.5: The initial data for the reduction game in the general case

Our first step in the reduction game is to determine to which multiplicity k the lines m_i are enforced by the Bezout Theorem to be contained in Γ. Since there are exactly 3 configuration points on each of these lines, the calculation is exactly the same as in (4.5). This means that Γ contains $\mathbb{M}$ with multiplicity at least $\frac{3}{2} \cdot 10^a + 1$. Removing this component from Γ we get

d	$[\mathbb{P}]$	$[\mathbb{Q}]$	$[\mathbb{Z}]$
$18 \cdot 10^a - 5$	$9 \cdot 10^a$	$6 \cdot 10^a - 2$	$1/2 \cdot 3 \cdot 10^a + 1$

By a slight abuse of notation, we call the resulting divisor again Γ in the hope that it will lead to no confusion. In the next step we consider the ℓ_i lines. Here we need to determine the least integer k such that

$$18 \cdot 10^a - 5 - k \geq 2(9 \cdot 10^a - k) + 2(6 \cdot 10^a - 2 - k)$$

holds. This amounts to $k = 4 \cdot 10^a + 1$. This means that Γ contains $\mathbb{L}$ with that multiplicity. Subtracting that much of $\mathbb{L}$ from Γ gives the following data for the new Γ curve.

d	$[\mathbb{P}]$	$[\mathbb{Q}]$	$[\mathbb{Z}]$
$6 \cdot 10^a - 8$	$10^a - 2$	$2 \cdot 10^a - 3$	$1/2 \cdot 3 \cdot 10^a + 1$

Similarly as in the special case, we are now in the position that we can step by step remove from Γ the divisor $2\mathbb{K} + 3\mathbb{M}$ so many times k that the degree of the curve and all multiplicities remain non-negative. It is easy to verify that these conditions are satisfied for $k = \frac{1}{4} \cdot 10^a - 1$ (this is an integer because we have $a \gg 1$). Applying this reduction step yields a curve Γ with the following data.

d	$[\mathbb{P}]$	$[\mathbb{Q}]$	$[\mathbb{Z}]$
16	2	5	7

This is exactly the data in the third bottom line of Table 4.4 in the special case and we conclude the general case in exactly the same manner as we did the special case.

It follows that S must contain the union of all 24 planes Π_i. Subtracting this union from S we get a surface whose degree and vanishing orders at points P_i, Q_j and Z_k have the same quotients as in the surface S. This implies that we can play the reduction game with that surface and then with the surface obtain from that surface by subtraction the union of Π_i planes and so on. Since the procedure never stops (we run ultimately in a contradiction with Bezout's Theorem) we get a contradiction to our assumption about the existence of S and we are done. $\square$

4.4.1 Resurgence of F_4

In this part we consider the resurgence of the F_4 configuration. We were not able to calculate its value but nevertheless we provide an upper bound and conjecture the actual value.

To begin with, we consider the *regularity* of the F_4 configuration. The following date is computer generated but it can be verified in dull hand calculation.

We have

$$0 \to R^4(-6) \oplus R(-8) \xrightarrow{\alpha} R^{16}(-5) \xrightarrow{\beta^T} R^{12}(-4) \xrightarrow{\gamma} \mathcal{I} \to 0,$$

where

$$\alpha = \begin{bmatrix} 0 & -z & 0 & -z & -\frac{1}{2}x^2z + \frac{1}{2}y^2z - \frac{1}{2}z^3 + \frac{1}{2}zw^2 \\ 0 & 0 & x & -x & xy^2 - xw^2 \\ -w & 0 & 0 & 0 & -xyz \\ 0 & 0 & -y & 0 & 0 \\ 0 & 0 & z & 0 & 0 \\ 0 & 0 & -x & 0 & \frac{1}{2}x^3 - \frac{1}{2}xy^2 - \frac{1}{2}xz^2 + \frac{1}{2}xw^2 \\ 0 & 0 & y & -y & \frac{1}{2}x^2y + \frac{1}{2}y^3 - \frac{1}{2}yz^2 - \frac{1}{2}yw^2 \\ x & 0 & 0 & 0 & yzw \\ 0 & 0 & -w & w & \frac{1}{2}x^2w - \frac{1}{2}y^2w - \frac{1}{2}z^2w + \frac{1}{2}w^3 \\ 0 & x & 0 & x & -xy^2 + xz^2 \\ -y & 0 & 0 & 0 & -xzw \\ z & 0 & 0 & 0 & xyw \\ 0 & -y & 0 & 0 & 0 \\ 0 & w & 0 & 0 & 0 \\ 0 & 0 & 0 & -z & 0 \\ 0 & 0 & 0 & w & 0 \end{bmatrix}$$

$$\beta = \begin{bmatrix}
0 & 0 & -x & y & 0 & 0 & w & 0 & 0 & 0 & 0 & 0 \\
y & -y & -z & 0 & z & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -z & 0 & -x & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-x & -x & 0 & -z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -x & 0 & -w & 0 & 0 & 0 & 0 & 0 \\
y & 0 & -z & 0 & 0 & z & 0 & w & 0 & 0 & 0 & 0 \\
-x & 0 & 0 & -z & 0 & y & 0 & 0 & w & 0 & 0 & 0 \\
0 & 0 & 0 & -w & 0 & 0 & -y & 0 & z & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -z & -x & y & 0 & 0 & 0 \\
0 & -y & -z & 0 & 0 & 0 & 0 & 0 & 0 & w & 0 & 0 \\
0 & 0 & 0 & 0 & -w & 0 & -x & 0 & 0 & z & 0 & 0 \\
0 & -w & 0 & 0 & 0 & 0 & 0 & 0 & -x & y & 0 & 0 \\
0 & -x & 0 & -z & 0 & 0 & 0 & 0 & 0 & 0 & w & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & z & 0 & 0 & -x & y & 0 \\
0 & 0 & x & 0 & -x & 0 & -w & 0 & 0 & 0 & 0 & w \\
0 & 0 & 0 & 0 & 0 & 0 & z & x & 0 & -x & 0 & z
\end{bmatrix}$$

and one row matrix

$$\begin{aligned}
\gamma = \quad & [z^3w - zw^3, & x^2zw - y^2zw, & \quad y^3w - yw^3, & xy^2w - xz^2w, \\
& x^2yw - yz^2w, & x^3w - xw^3, & \quad xyz^2 - xyw^2, & y^3z - yz^3, \\
& xy^2z - xzw^2, & x^2yz - yzw^2, & \quad x^3z - xz^3, & x^3y - xy^3]
\end{aligned}$$

and

$$\operatorname{reg}(\mathcal{I}) = \max\{4-0, 5-1, 8-2, 6-2\} = 6.$$

Revoking Theorem 3.10 we obtain

$$\frac{3}{2} \leq \rho(\mathcal{I}) \leq \frac{9}{4}.$$

The computer experiments we were able to carry out provided the data displayed in Table 4.6.

r	1	2	3	4	5	6	7	8	9	10
m	1	2	3	5	7	8	9	11	13	14
m/r	1	1	1	1.25	1.4	1.33	1.28	1.37	1.44	1.4

Table 4.6: Non-containments $\mathcal{I}^{(m)} \not\subset \mathcal{I}^r$ for small values of r and m.

That data is so to interpret that for each r, the value of m is taken so that

$$\mathcal{I}^{(m)} \not\subset \mathcal{I}^r \text{ but } \mathcal{I}^{(m+1)} \subset \mathcal{I}^r.$$

Motivated by Table 4.6 we conclude this section with the following conjecture.

Conjecture 4.16
The resurgence of the F_4 configuration is $\frac{3}{2}$.

4.5 A 20 points subset of F_4

Here we study a configuration of points which does not come from any root system but nevertheless fits nicely between the D_4 and F_4 configurations. It is interesting in the context of our work because it is an instance where not only linear subspaces need to be considered in order to compute the Waldschmidt constant. Specifically, we consider the following set Z_{20} of 20 points.

Definition 4.17
Let Z_{20} be a configuration which contains all 12 points from D_4 and additional 8 points from F_4, with the following coordinates:

$$\begin{array}{lll}
P_1 = [1:-1:0:0] & P_2 = [0:1:-1:0] & P_3 = [0:0:1:-1] \\
P_4 = [0:0:1:1] & P_5 = [1:0:-1:0] & P_6 = [0:1:0:-1] \\
P_7 = [0:1:0:1] & P_8 = [1:0:0:-1] & P_9 = [1:0:0:1] \\
P_{10} = [0:1:1:0] & P_{11} = [1:0:1:0] & P_{12} = [1:1:0:0] \\
P_{13} = [1:1:1:1] & P_{14} = [-1:-1:1:1] & P_{15} = [-1:1:1:-1] \\
P_{16} = [-1:1:-1:1] & P_{17} = [1:1:1:-1] & P_{18} = [1:1:-1:1] \\
P_{19} = [1:-1:1:1] & P_{20} = [1:-1:-1:-1] &
\end{array}$$

Consider the following quadrics:

$$\begin{aligned}
Q_1 &: x^2 - y^2 + z^2 - w^2 = 0, \\
Q_2 &: x^2 + y^2 - z^2 - w^2 = 0, \\
Q_3 &: x^2 - y^2 - z^2 + w^2 = 0.
\end{aligned}$$

Every quadric Q_i contains 16 points from Z_{20}. In the following table, we present how the points are distributed on the quadrics.

$$\begin{array}{lcccccccc} Q_1: & P_1 & P_2 & P_3 & P_4 & P_8 & P_9 & P_{10} & P_{12} \\ & P_{13} & P_{14} & P_{15} & P_{16} & P_{17} & P_{18} & P_{19} & P_{20} \\ Q_2: & P_2 & P_5 & P_6 & P_7 & P_8 & P_9 & P_{10} & P_{11} \\ & P_{13} & P_{14} & P_{15} & P_{16} & P_{17} & P_{18} & P_{19} & P_{20} \\ Q_3: & P_1 & P_3 & P_4 & P_5 & P_6 & P_7 & P_{11} & P_{12} \\ & P_{13} & P_{14} & P_{15} & P_{16} & P_{17} & P_{18} & P_{19} & P_{20}. \end{array}$$

Additionally consider the following 8 planes:

$$\begin{array}{ll} \Pi_1 : x - y + z + w = 0 & \Pi_2 : x + y + z + w = 0 \\ \Pi_3 : x - y + z - w = 0 & \Pi_4 : x + y + z - w = 0 \\ \Pi_5 : x - y - z + w = 0 & \Pi_6 : x + y - z + w = 0 \\ \Pi_7 : x - y - z - w = 0 & \Pi_8 : x + y - z - w = 0. \end{array}$$

Every plane Π_i contains 9 points from Z_{20}. Incidences between these planes and the points in Z_{20} are presented next:

$$\begin{array}{llllllllll} \Pi_1: & P_3 & P_5 & P_7 & P_8 & P_{10} & P_{12} & P_{17} & P_{18} & P_{20} \\ \Pi_2: & P_1 & P_2 & P_3 & P_5 & P_6 & P_8 & P_{14} & P_{15} & P_{16} \\ \Pi_3: & P_4 & P_5 & P_6 & P_9 & P_{10} & P_{12} & P_{13} & P_{14} & P_{15} \\ \Pi_4: & P_1 & P_2 & P_4 & P_5 & P_7 & P_9 & P_{17} & P_{18} & P_{20} \\ \Pi_5: & P_2 & P_4 & P_7 & P_8 & P_{11} & P_{12} & P_{13} & P_{14} & P_{16} \\ \Pi_6: & P_1 & P_4 & P_6 & P_8 & P_{10} & P_{11} & P_{17} & P_{19} & P_{20} \\ \Pi_7: & P_2 & P_3 & P_6 & P_9 & P_{11} & P_{12} & P_{17} & P_{18} & P_{19} \\ \Pi_8: & P_1 & P_3 & P_7 & P_9 & P_{10} & P_{11} & P_{13} & P_{15} & P_{16} \end{array}$$

We aim to establish a connection between points on the quadrics Q_i and the rulings on each quadric. For this purpose, it is advantageous to employ the natural parametrization of a smooth quadric in $\mathbb{P}^3$ through a suitable Segre map from $\mathbb{P}^1 \times \mathbb{P}^1$. We provide specific parametrizations for each quadric Q_i and explicitly identify the preimages of points from Z_{20} on the factors $\mathbb{P}^1 \times \mathbb{P}^1$. To avoid confusion, we acknowledge that, by a slight abuse of notation, we employ the same symbols to denote points in Z_{20} and their preimages.

The quadric Q_1 is parametrized by

$$(s : t) \times (u : v) \to (su + tv : tv - su : tu - sv : sv + tu),$$

so that points in $Z_{20} \cap Q_1$ can be identified with points on $\mathbb{P}^1 \times \mathbb{P}^1$ in the following manner:

$$P_1 = (1, 0) \times (1, 0) \qquad P_2 = (1, 1) \times (1, -1)$$

$$P_3 = (1,0) \times (0,1) \qquad P_4 = (0,1) \times (1,0)$$

$$P_8 = (-1,1) \times (-1,1) \qquad P_9 = (1,1) \times (1,1)$$

$$P_{10} = (1,-1) \times (1,1) \qquad P_{12} = (0,1) \times (0,1)$$

$$P_{13} = (0,1) \times (1,1) \qquad P_{14} = (1,0) \times (1,-1)$$

$$P_{15} = (1,0) \times (1,1) \qquad P_{16} = (1,0) \times (-1,1)$$

$$P_{17} = (1,-1) \times (0,-1) \qquad P_{18} = (1,1) \times (0,1)$$

$$P_{19} = (1,1) \times (1,0) \qquad P_{20} = (-1,1) \times (1,0).$$

The distribution of these points related to the product structure on $\mathbb{P}^1 \times \mathbb{P}^1$ is presented in Figure 4.7.

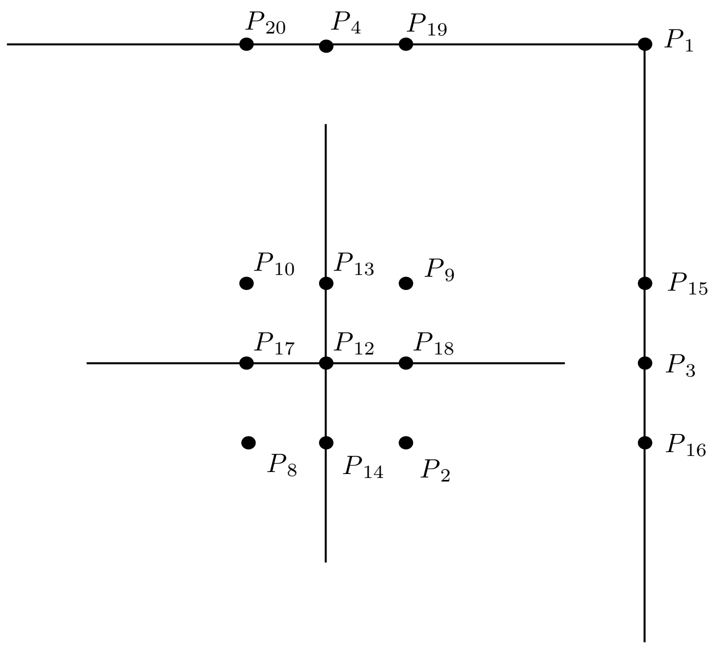


Figure 4.7: Quadric Q_1

Passing to Q_2 its parametrization is provided by

$$(s:t) \times (u:v) \to (su + tv : tu - sv : tv - su : sv + tu).$$

The points in $Z_{20} \cap Q_2$ are then:

$$P_2 = (1,1) \times (-1,1) \qquad P_5 = (1,0) \times (1,0)$$

$$P_6 = (1,0) \times (0,1) \qquad P_7 = (0,1) \times (1,0)$$

$$P_8 = (-1,1) \times (-1,1) \qquad P_9 = (1,1) \times (1,1)$$

$$P_{10} = (-1, 1) \times (1, 1) \qquad P_{11} = (0, 1) \times (0, 1)$$
$$P_{13} = (0, 1) \times (1, 1) \qquad P_{14} = (1, 1) \times (0, 1)$$
$$P_{15} = (1, 1) \times (1, 0) \qquad P_{16} = (-1, 1) \times (0, 1)$$
$$P_{17} = (1, 0) \times (-1, 1) \qquad P_{18} = (0, 1) \times (-1, 1)$$
$$P_{19} = (1, 0) \times (1, 1) \qquad P_{20} = (-1, 1) \times (1, 0).$$

The distribution of these points related to the product structure on $\mathbb{P}^1 \times \mathbb{P}^1$ is presented in Figure 4.8.

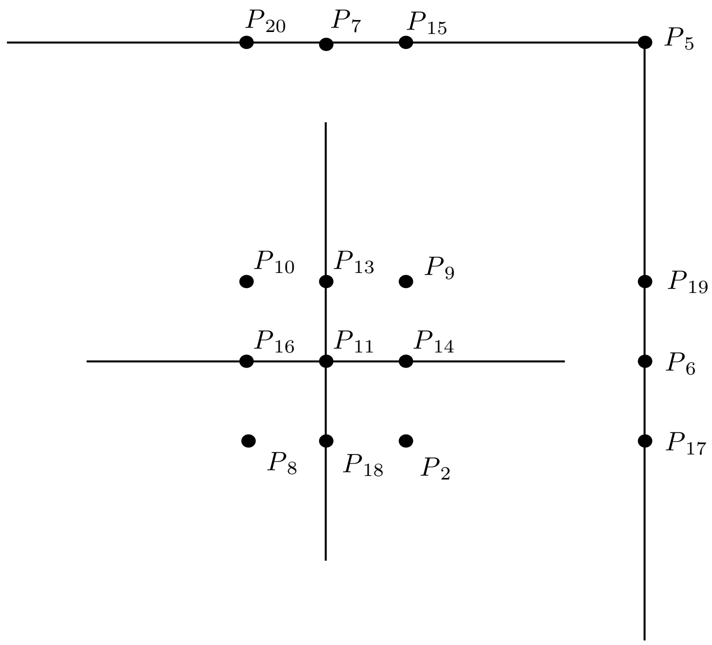


Figure 4.8: Quadric Q_2

Finally we consider Q_3 parametrized by the mapping

$$(s, t) \times (u, v) \to (su + tv : tv - su : sv + tu : tu - sv).$$

The points in $Z_{20} \cap Q_3$ have then coordinates:

$$P_1 = (1, 0) \times (1, 0) \qquad P_3 = (1, 0) \times (0, 1)$$
$$P_4 = (0, 1) \times (1, 0) \qquad P_5 = (-1, 1) \times (-1, 1)$$
$$P_6 = (1, 1) \times (-1, 1) \qquad P_7 = (-1, 1) \times (1, 1)$$
$$P_{11} = (1, 1) \times (1, 1) \qquad P_{12} = (0, 1) \times (0, 1)$$
$$P_{13} = (0, 1) \times (1, 1) \qquad P_{14} = (1, 1) \times (1, 0)$$

$$P_{15} = (-1,1) \times (0,1) \qquad P_{16} = (1,1) \times (0,1)$$
$$P_{17} = (0,1) \times (-1,1) \qquad P_{18} = (1,0) \times (1,1)$$
$$P_{19} = (1,0) \times (-1,1) \qquad P_{20} = (-1,1) \times (1,0).$$

The distribution of these points related to the product structure on $\mathbb{P}^1 \times \mathbb{P}^1$ is presented in Figure 4.9.

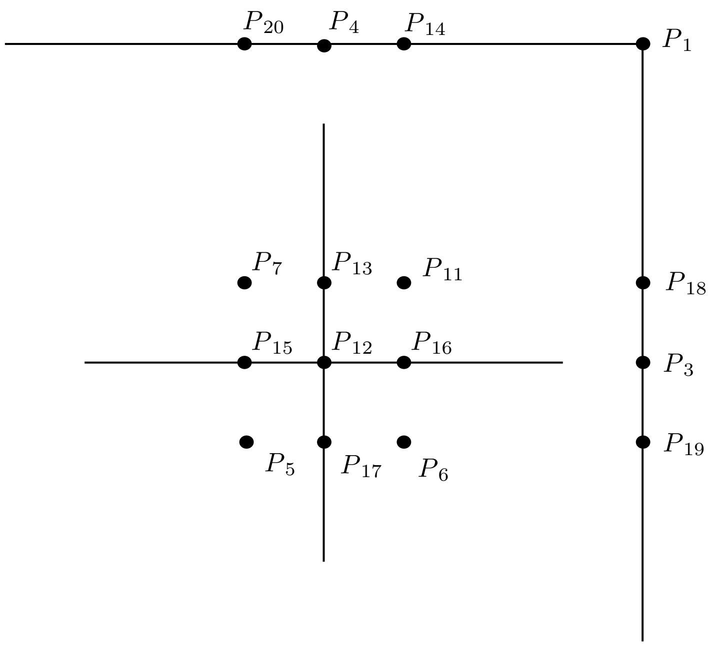


Figure 4.9: Quadric Q_3

It follows immediately that points from Z_{20} in each of the quadrics Q_i form a $(4,4)$-grid. Additionally, we observe that the points on each grid line are harmonic. To check this claim, it suffices, by symmetry, to compute just one cross-ratio. We consider the points P_1, P_{14}, P_4, P_{20} and obtain with $p = 1, q = -1$:

$$[1 : -1 : 0 : 0] = p[1 : -1 : 1 : 1] + q[0 : 0 : 1 : 1].$$

Similarly, with $r = 1, s = -2$ we get

$$[1 : -1 : -1 : -1] = r[1 : -1 : 1 : 1] + s[0 : 0 : 1 : 1].$$

Consequently, the cross-ratio is equal to

$$\frac{-1 \cdot 1}{1 \cdot (-2)} = \frac{1}{2},$$

so the points are harmonic.

Theorem 4.18
Waldschmidt constant for Z_{20} is equal to $\frac{7}{3}$.

PROOF. Our proof here goes along the same lines as the proof of Theorem 4.15, where we computed the Waldschmidt constant of the F_4 configuration. In fact the proof is much easier, because the divisor

$$D = \Pi_1 + \ldots + \Pi_8 + Q_1 + Q_2 + Q_3$$

of degree 14 vanishes at all points of Z_{20} to the same order 6. This provides an upper bound for $\widehat{\alpha}(Z_{20})$.

To see that it is also a lower bound, assume to the contrary that there exists a surface S of degree d vanishing to multiplicity at least m at all points of Z_{20} with

$$\frac{d}{m} < \frac{7}{3}.$$

Restricting S to every component of D, it is easy to show that S must contain every component D as its own component. So there is a residual surface S' of degree $d - 14$ vanishing at all points of Z_{20} to order at least $m - 6$. Since

$$\frac{d-14}{m-6} < \frac{7}{3},$$

we can repeat the argument that S' contains D and continuing in this manner we arrive to a contradiction.

□

We continue by considering the *resurgence* of the set of 20 points as above. We have the following result.

Proposition 4.19
There is

$$\rho(\mathcal{I}(Z_{20})) = \frac{12}{7}.$$

PROOF. To begin with we note that the regularity of $\mathcal{I} = \mathcal{I}(Z_{20})$ is 4.

Taking this for granted for the moment, the claim of the Proposition follows immediately by Theorem 3.10. Indeed, we have

$$\frac{12}{7} = \frac{\alpha(\mathcal{I})}{\widehat{\alpha}(\mathcal{I})} \leqslant \frac{\operatorname{reg}(\mathcal{I})}{\widehat{\alpha}(\mathcal{I})} = \frac{12}{7},$$

Turning back to the regularity, we have the following resolution of $\mathcal{I}$

$$0 \to R^{10}(-6) \xrightarrow{\alpha} R^{24}(-5) \xrightarrow{\beta^T} R^{15}(-4) \xrightarrow{\gamma} \mathcal{I} \to 0.$$

The matrices of the maps appearing in the resolution have the following form:

$$\alpha = \begin{bmatrix}
0 & -z & 0 & -z & w & 0 & 0 & 0 & x & 0 \\
0 & 0 & x & -x & 0 & 0 & 0 & y & 0 & -w \\
-w & 0 & 0 & 0 & 0 & 0 & -x & -z & 0 & 0 \\
0 & 0 & -y & 0 & 0 & w & 0 & -x & 0 & 0 \\
0 & 0 & z & 0 & w & 0 & 0 & 0 & -x & 0 \\
0 & 0 & -x & 0 & 0 & 0 & 0 & -y & z & 0 \\
0 & 0 & y & -y & 0 & 0 & -z & x & 0 & 0 \\
x & 0 & 0 & 0 & y & 0 & w & 0 & 0 & 0 \\
0 & 0 & -w & w & -z & 0 & 0 & 0 & 0 & x \\
0 & x & 0 & x & 0 & 0 & 0 & -y & -z & 0 \\
-y & 0 & 0 & 0 & -x & 0 & 0 & 0 & w & 0 \\
z & 0 & 0 & 0 & 0 & -x & 0 & w & 0 & 0 \\
0 & -y & 0 & 0 & 0 & 0 & z & x & 0 & 0 \\
0 & w & 0 & 0 & -z & 0 & 0 & 0 & 0 & -x \\
0 & 0 & 0 & -z & 0 & 0 & -y & 0 & x & 0 \\
0 & 0 & 0 & w & 0 & -y & 0 & 0 & 0 & x \\
0 & 0 & 0 & 0 & x & 0 & 0 & 0 & 0 & -z \\
0 & 0 & 0 & 0 & y & 0 & -w & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & x & 0 & 0 & 0 & y \\
0 & 0 & 0 & 0 & 0 & z & w & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -z & -y & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & x & 0 & y & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & w & 0 & -y \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w & z
\end{bmatrix}$$

$$\beta = \begin{bmatrix}
0 & 0 & -x & y & 0 & 0 & w & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
y & -y & -z & 0 & z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -z & 0 & -x & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-x & -x & 0 & -z & 0 & z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -x & y & -w & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
y & 0 & -z & 0 & 0 & 0 & 0 & 0 & w & 0 & 0 & 0 & 0 & 0 & 0 \\
-x & 0 & 0 & -z & 0 & 0 & 0 & 0 & 0 & w & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -w & 0 & 0 & -y & 0 & 0 & z & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -z & 0 & -x & y & 0 & 0 & 0 & 0 & 0 \\
0 & -y & -z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -w & 0 & -x & 0 & 0 & 0 & z & 0 & 0 & 0 & 0 \\
0 & -w & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x & y & 0 & 0 & 0 & 0 \\
0 & -x & 0 & -z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & z & 0 & 0 & 0 & -x & y & 0 & 0 & 0 \\
0 & 0 & x & 0 & -x & 0 & -w & 0 & 0 & 0 & 0 & 0 & 0 & w & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & z & 0 & x & 0 & -x & 0 & 0 & z & 0 \\
0 & 0 & w & 0 & 0 & 0 & -x & y & -z & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -w & y & -x & 0 & 0 & 0 & z & 0 & 0 & 0 \\
w & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y & -x & 0 & 0 & z & 0 & 0 \\
0 & 0 & 0 & w & 0 & -w & y & 0 & 0 & 0 & 0 & 0 & -x & y & 0 \\
0 & z & y & 0 & 0 & -x & 0 & -w & 0 & 0 & 0 & 0 & 0 & 0 & w \\
-z & 0 & y & -x & 0 & 0 & 0 & -w & 0 & 0 & 0 & 0 & w & 0 & 0 \\
0 & w & 0 & 0 & 0 & 0 & 0 & -z & y & 0 & 0 & -x & 0 & 0 & z \\
0 & 0 & 0 & 0 & w & 0 & 0 & 0 & -z & 0 & 0 & 0 & -y & -x & y \\
0 & 0 & 0 & 0 & w & 0 & 0 & 0 & -z & 0 & 0 & 0 & -y & -x & y
\end{bmatrix}$$

and a row vector of all 15 generators of $\mathcal{I}$:

$$\begin{aligned}
\gamma = \quad & [z^3w - zw^3, && x^2zw - y^2zw, && y^3w - yw^3, \\
& xy^2w - xz^2w, && x^2yw - yz^2w, && x^3w - xw^3, \\
& xyz^2 - xyw^2, && x^2z^2 + y^2z^2 - z^4 - x^2w^2 - y^2w^2 + w^4, \\
& y^3z - yz^3, && xy^2z - xzw^2, && x^2yz - yzw^2, \\
& x^3z - xz^3, && x^2y^2 - y^4 + y^2z^2 - x^2w^2 - z^2w^2 + w^4, \\
& x^3y - xy^3, && x^4 - y^4 + y^2z^2 - z^4 - x^2w^2 + w^4]
\end{aligned}$$

It follows that the regularity of the ideal of Z_{20} is 4. $\square$

4.6 The H_4 root system

In this part we describe the highly symmetric configuration, namely the H_4 configuration coming from the H_4 root system. This system is a little bit different from those considered so far, because it is not crystalographic.

Definition 4.20

Let $Z(H_4)$ be the set of 60 points, which in suitable coordinates can be take as in the following list:

$$\begin{array}{lll}
P_1 = [1:0:0:0]\,, & P_2 = [0:1:0:0]\,, & P_3 = [0:0:1:0]\,, \\
P_4 = [0:0:0:1]\,, & P_5 = [1:1:1:1]\,, & P_6 = [1:1:1:-1]\,, \\
P_7 = [1:1:-1:1]\,, & P_8 = [1:1:-1:-1]\,, & P_9 = [1:-1:1:1]\,, \\
P_{10} = [1:-1:1:-1]\,, & P_{11} = [1:-1:-1:1]\,, & P_{12} = [1:-1:-1:-1]\,, \\
P_{13} = [0:\varphi:\varphi^2:1]\,, & P_{14} = [0:\varphi:\varphi^2:-1]\,, & P_{15} = [0:\varphi:-\varphi^2:1]\,, \\
P_{16} = [0:\varphi:-\varphi^2:-1]\,, & P_{17} = [0:\varphi^2:1:\varphi]\,, & P_{18} = [0:\varphi^2:1:-\varphi]\,, \\
P_{19} = [0:\varphi^2:-1:\varphi]\,, & P_{20} = [0:\varphi^2:-1:-\varphi]\,, & P_{21} = [0:1:\varphi:\varphi^2]\,, \\
P_{22} = [0:1:\varphi:-\varphi^2]\,, & P_{23} = [0:1:-\varphi:\varphi^2]\,, & P_{24} = [0:1:-\varphi:-\varphi^2]\,, \\
P_{25} = [\varphi:0:1:\varphi^2]\,, & P_{26} = [\varphi:0:1:-\varphi^2]\,, & P_{27} = [\varphi:0:-1:\varphi^2]\,, \\
P_{28} = [\varphi:0:-1:-\varphi^2]\,, & P_{29} = [\varphi^2:0:\varphi:1]\,, & P_{30} = [\varphi^2:0:\varphi:-1]\,, \\
P_{31} = [\varphi^2:0:-\varphi:1]\,, & P_{32} = [\varphi^2:0:-\varphi:-1]\,, & P_{33} = [1:0:\varphi^2:\varphi]\,, \\
P_{34} = [1:0:\varphi^2:-\varphi]\,, & P_{35} = [1:0:-\varphi^2:\varphi]\,, & P_{36} = [1:0:-\varphi^2:-\varphi]\,, \\
P_{37} = [\varphi:\varphi^2:0:1]\,, & P_{38} = [\varphi:\varphi^2:0:-1]\,, & P_{39} = [\varphi:-\varphi^2:0:1]\,, \\
P_{40} = [\varphi:-\varphi^2:0:-1]\,, & P_{41} = [\varphi^2:1:0:\varphi]\,, & P_{42} = [\varphi^2:1:0:-\varphi]\,, \\
P_{43} = [\varphi^2:-1:0:\varphi]\,, & P_{44} = [\varphi^2:-1:0:-\varphi]\,, & P_{45} = [1:\varphi:0:\varphi^2]\,, \\
P_{46} = [1:\varphi:0:-\varphi^2]\,, & P_{47} = [1:-\varphi:0:\varphi^2]\,, & P_{48} = [1:-\varphi:0:-\varphi^2]\,, \\
P_{49} = [\varphi:1:\varphi^2:0]\,, & P_{50} = [\varphi:1:-\varphi^2:0]\,, & P_{51} = [\varphi:-1:\varphi^2:0]\,, \\
P_{52} = [\varphi:-1:-\varphi^2:0]\,, & P_{53} = [\varphi^2:\varphi:1:0]\,, & P_{54} = [\varphi^2:\varphi:-1:0]\,, \\
P_{55} = [\varphi^2:-\varphi:1:0]\,, & P_{56} = [\varphi^2:-\varphi:-1:0]\,, & P_{57} = [1:\varphi^2:\varphi:0]\,, \\
P_{58} = [1:\varphi^2:-\varphi:0]\,, & P_{59} = [1:-\varphi^2:\varphi:0]\,, & P_{60} = [1:-\varphi^2:-\varphi:0]\,,
\end{array}$$

where φ denotes the golden ratio. We refer to this set of points as the H_4 *configuration of points.*

The choice of particular coordinates does not influence our considerations as the following Remark explains.

Remark 4.21

Up to a projective change of coordinates, there is just one way to embed H_4 into $\mathbb{P}^3$, see [1].

The set Z_{H_4} determines 60 dual planes $V_1, \ldots, V_{60}$, which we consider in the same $\mathbb{P}^3$ rather than in the dual space. More specifically, for $P_i = [a:b:c:d]$ we define $V_i = \{ax + by + cz + dw = 0\}$.

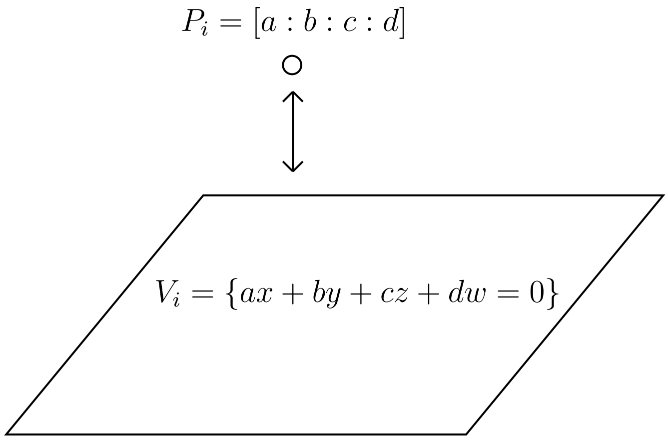


Figure 4.10: Dual plane

These points and planes form a symmetric (60_{15})-configuration of points and planes, which means that there are 15 points in each plane and 15 planes pass through each point. We call the planes V_i the 15-reach planes to indicate that they contain 15 points from the H_4 configuration. Specific incidences between the configuration points and the 15-reach planes are listed in Table 5.2 in which the points are represented by numbers only.

In each of these planes, the 15 distinguished points form an H_3 configuration, which was presented previously. This is indicated in Figure 4.1 in affine coordinates (x, y), where $\{z = 0\}$ is the line at the infinity.

Additional collinearities, which are of interest to us, determine 72 lines which contain exactly 5 configuration points (and which we call 5-reach lines for this reason). Moreover, 5 is the maximal number of collinear points in the configuration.

Through each point of configuration there are exactly six 5-reach lines passing. The incidences are indicated in Table 5.3 where points are represented again by numbers only.

Because of the duality between points P_i and planes V_i, each of the 5-reach lines is contained in exactly 5 of the 15-reach planes and each of these planes contain exactly six 5-reach lines.

One could naively expect that the 60 planes dual to the 60 points in Z_{H_4} provide a divisor computing the Waldschmidt constant $\widehat{\alpha}(Z_{H_4})$. This is however not true for very fundamental reasons which we state now.

Lemma 4.22

Let Z be a set of s points in $\mathbb{P}^N$. Then

$$\widehat{\alpha}(Z) \leq \sqrt[N]{s}.$$

PROOF. We sketch a proof for lack of reference. There is the following bound

$$\alpha(I_Z^{(m)}) \leq d(m),$$

where $d(m)$ is the least integer d such that

$$\binom{d+N}{N} > s\binom{m+N-1}{N}. \tag{4.6}$$

Comparing the leading terms on both sides of (4.6) (in d on the left and in m on the right), we see that the inequality holds for large m and d provided $d > m\sqrt[N]{s}$. Since our statement is asymptotic, we may take $d(m) = \lfloor m\sqrt[N]{s} + 1 \rfloor$ and the claim follows.

$\square$

Now, the arrangement of 60 planes dual to points in the H_4 configuration gives

$$\alpha(I(Z_{H_4})^{(15)}) \leq 60,$$

hence

$$\widehat{\alpha}(Z_{H_4}) \leq 4.$$

But from Lemma 4.22 we derive immediately the following consequence.

Corollary 4.23

For the Waldschmidt constant of Z_{H_4} we have

$$\widehat{\alpha}(Z_{H_4}) \leq \sqrt[3]{60} \cong 3.9.$$

However, using symbolic computations we were able to detect 4 generators of degree 19 in $I(Z_{H_4})^{(15)}$, so that we have.

Theorem 4.24

The Waldschmidt constant of the H_4 configuration of points satisfies

$$\widehat{\alpha}(Z_{H_4}) \leq \frac{19}{5} = 3.8.$$

We were not able neither to explain the existence of the aforementioned surfaces of degree 19 with singularities of multiplicity at least 5 at all points of H_4, nor could we proof that the actual value of the Waldschmidt constant is not strictly less than 3.8. Thus the determination of the exact value of the Waldschmidt constant for the H_4 configuration remains an intriguing open problem.

5 Properties of general projections of symmetric sets of points in projective spaces

In the last section, we change the course of our research and focus on properties of general projections a symmetric set of points in projective spaces. This part grew out from a study paralleling the main subject of this thesis and was carried out with Maciej Zięba. I am grateful to him for allowing me to include our results in my thesis.

The motivation for this part of our work comes from Polizzi's question asked over 12 years ago on Math-Overflow (see: https://mathoverflow.net/questions/67265/when-is-a-general-projection-of-d2-points-in-mathbbp3-a-complete-inters): *Do there exist sets of points Z in $\mathbb{P}^3$ such that the general projection of Z to $\mathbb{P}^2$ is a complete intersection?*

The following notion was introduced in [26].

Definition 5.1 (*Geproci*)
We say that a finite set $Z \subset \mathbb{P}^3$ has a *geproci* property (the acronim comes from: GEneral PROjection is a Complete Intersection), if its projection from a general point in $\mathbb{P}^3$ to $\mathbb{P}^2$ is a complete intersection.

In 2011, Dmitri Panov answered the above question pointing out that grids have the property stipulated by Polizzi. Two remarks are here in place.

Remark 5.2
An (a, b)-grid is non-degenerate for $b \geq a \geq 2$.

Remark 5.3
An (a, b)-grid has the geproci property.

After Panov's answer Polizzi edited his post and asked:

1. *Are there other (than grids) configurations of points with the same property?*

2. *It is possible to classify them up to projective transformations (at least for small numbers of points)?*

These questions have been considered in Levico Terme in 2018 during the workshop on Lefschetz Properties and Jordan Type in Algebra, Geometry and Combinatorics by A. Bernardi, L. Chiantini, G. Denham, G. Favacchio, B. Harbourne, J. Migliore, T. Szemberg and J. Szpond. This working group considered unexpected surfaces. In particular they considered the F_4 configuration and observed that this root system does not form a $(4,6)$–*grid* but its general projection is a complete intersection of type $(4,6)$.

The set F_4 is contained in 6 disjoint lines:

$$\begin{aligned} z = w = 0, \qquad & x = y = 0, \qquad & x - z = y - w = 0, \\ x + z = y + w = 0, \quad & x - w = y + z = 0, \quad & x + w = y - z = 0 \end{aligned}$$

and there are 4 configuration points on each of these lines.

During the same workshop in Levico Terme the working group considered the root system D_4 from the same perspective. In this case the general projection is also a complete intersection and this sets of 12 points is not a $(3,4)$–*grid*, (see [23]).

In 2020 Pokora, Szemberg and Szpond introduced the notion of *half-grids*, [26].

Definition 5.4 (*Half-grid*)
Let a and b be positive integers. A set Z of $a \cdot b$ points in $\mathbb{P}^3$ is an (a,b)-*half-grid* if there exists a set of mutually skew lines $L_1, \ldots, L_a$ covering Z and a general projection of Z to a hyperplane is a complete intersection of images of the lines with a (possibly reducible) curve of degree b.

This situation is visualized in Figure 5.1.:

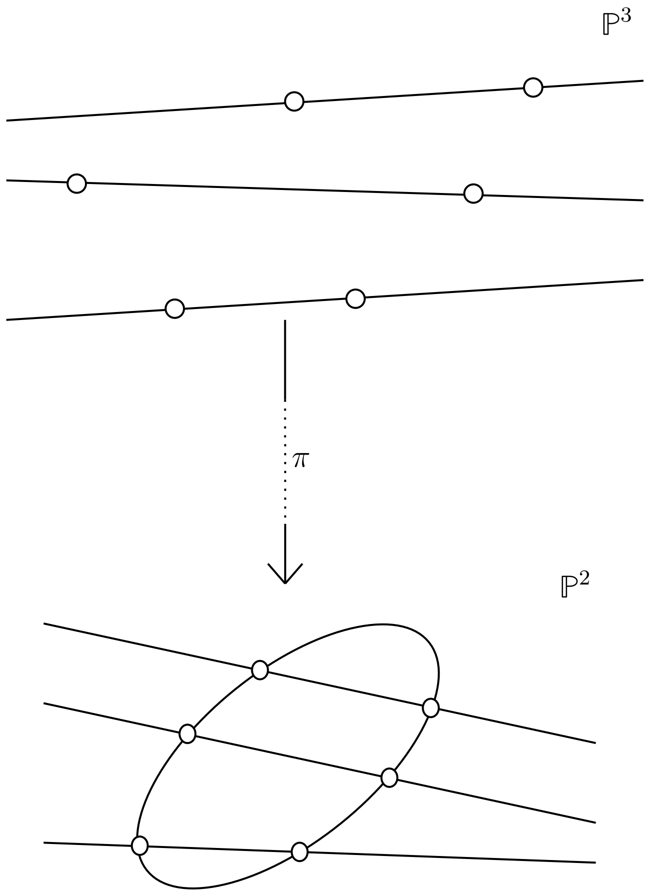


Figure 5.1: Half-grid

The three authors mentioned above found an interesting set of 60 points in $\mathbb{P}^3$, which, like F_4, is a half-grid rather than a grid. In particular they showed with their example than not all geproci sets are defined by root systems.

Definition 5.5

The set Z_{60} is called Klein configuration, which can be assigned the following coordinates:

$P_1 = [0:0:1:1]$	$P_2 = [0:0:1:i]$	$P_3 = [0:0:1:-1]$
$P_4 = [0:0:1:-i]$	$P_5 = [0:1:0:1]$	$P_6 = [0:1:0:i]$
$P_7 = [0:1:0:-1]$	$P_8 = [0:1:0:-i]$	$P_9 = [0:1:1:0]$
$P_{10} = [0:1:i:0]$	$P_{11} = [0:1:-1:0]$	$P_{12} = [0:1:-i:0]$
$P_{13} = [1:0:0:1]$	$P_{14} = [1:0:0:i]$	$P_{15} = [1:0:0:-1]$
$P_{16} = [1:0:0:-i]$	$P_{17} = [1:0:1:0]$	$P_{18} = [1:0:i:0]$
$P_{19} = [1:0:-1:0]$	$P_{20} = [1:0:-i:0]$	$P_{21} = [1:1:0:0]$
$P_{22} = [1:i:0:0]$	$P_{23} = [1:-1:0:0]$	$P_{24} = [1:-i:0:0]$
$P_{25} = [1:0:0:0]$	$P_{26} = [0:1:0:0]$	$P_{27} = [0:0:1:0]$

$$
\begin{array}{lll}
P_{28} = [0:0:0:1] & P_{29} = [1:1:1:1] & P_{30} = [1:1:1:-1] \\
P_{31} = [1:1:-1:1] & P_{32} = [1:1:-1:-1] & P_{33} = [1:-1:1:1] \\
P_{34} = [1:-1:1:-1] & P_{35} = [1:-1:-1:1] & P_{36} = [1:-1:-1:-1] \\
P_{37} = [1:1:i:i] & P_{38} = [1:1:i:-i] & P_{39} = [1:1:-i:i] \\
P_{40} = [1:1:-i:-i] & P_{41} = [1:-1:i:i] & P_{42} = [1:-1:i:-i] \\
P_{43} = [1:-1:-i:i] & P_{44} = [1:-1:-i:-i] & P_{45} = [1:i:1:i] \\
P_{46} = [1:i:1:-i] & P_{47} = [1:-i:1:i] & P_{48} = [1:-i:1:-i] \\
P_{49} = [1:i:-1:i] & P_{50} = [1:i:-1:-i] & P_{51} = [1:-i:-1:i] \\
P_{52} = [1:-i:-1:-i] & P_{53} = [1:i:i:1] & P_{54} = [1:i:-i:1] \\
P_{55} = [1:-i:i:1] & P_{56} = [1:-i:-i:1] & P_{57} = [1:i:i:-1] \\
P_{58} = [1:i:-i:-1] & P_{59} = [1:-i:i:-1] & P_{60} = [1:-i:-i:-1]\ .
\end{array}
$$

Theorem 5.6

The Klein configuration is $(6,10)$*-geproci.*

Inspired by this result, they asked if all geproci sets of points in $\mathbb{P}^3$ are half-grids. Establishing such a fact would provide a major step towards the classification of all geproci sets of points in $\mathbb{P}^3$. Somewhat disappointingly, we show that this is not the case. Our main result is thus the following.

Theorem 5.7

The H_4 *configuration of points in* $\mathbb{P}^3$ *has the geproci property and it is neither a half-grid nor a grid.*

PROOF. We begin by showing that H_4 root system has the geproci property. More precisely we will show that a general projection of H_4 to $\mathbb{P}^2$ is a $(6,10)$ complete intersection.

To this end let P be a general point in $\mathbb{P}^3$. Thus, in particular, P is not contained in any of the flats described in previous chapter. Let

$$\pi_P : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$$

be the projection from P.

The existence of a unique cone of degree 6 with vertex at P vanishing along the set H_4 was established in [23, Section 3.7]. We checked, using our Singular script, that this cone is smooth apart from its vertex. It implies that $\pi_P(H_4)$ is contained in a unique smooth curve C_6 of degree 6. It is visualised at Figure 5.2

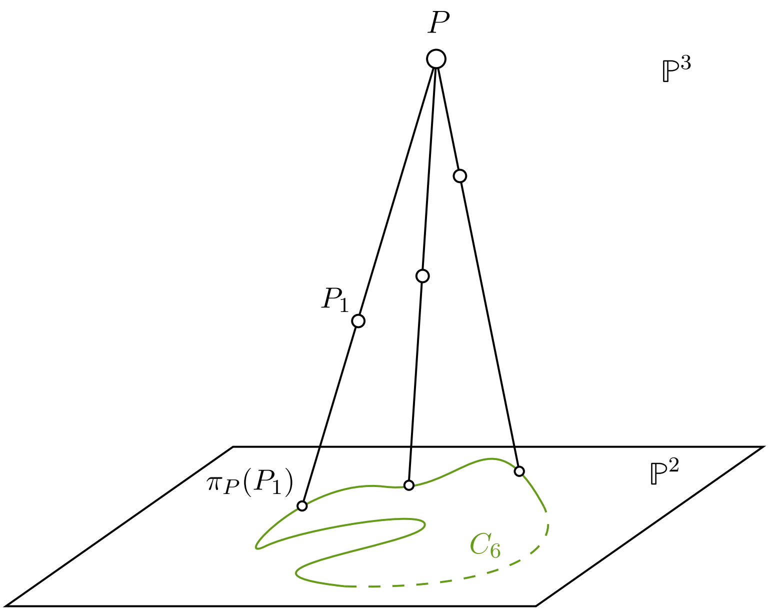


Figure 5.2: Projection

Thus, it remains to establish the existence of a curve of degree 10, not containing C_6 as a component, and vanishing at all points of $\pi_P(H_4)$. In the course we will reveal additional nice properties of the set H_4.

To begin with note that the following two sets of lines:

$$L_1 = \ell_1,\ L_2 = \ell_{25},\ L_3 = \ell_{32},\ L_4 = \ell_{37},\ L_5 = \ell_{44},$$
$$M_1 = \ell_2,\ M_2 = \ell_{26},\ M_3 = \ell_{31},\ M_4 = \ell_{38},\ M_5 = \ell_{43}$$

form a $(5,5)$-grid which consists of 25 points with the following numbers:

$$1, 5, 6, 7, 8, 13, 14, 15, 16, 29, 30, \ldots, 38, 41, 42, 51, 52, 57, 58.$$

It was established in [8, Remark 3.4] that all (a,b)-grids with $a, b \geq 3$ are contained in a quadric. In our case the equation of this quadric is:

$$Q_1 :\ 2xy - y^2 + (\varphi - 1)z^2 - \varphi w^2 = 0.$$

Let g_i be the equation of the plane generated by L_i and P with $i \in \{1, ..., 5\}$. Similarly, denote by h_i the equation of the plane spanned by M_i and P with $i \in \{1, ..., 5\}$. Then the products

$$g = g_1 \cdot \ldots \cdot g_5 \ \text{ and } \ h = h_1 \cdot \ldots \cdot h_5$$

generate a pencil of cones of degree 5 with vertex at P vanishing at all points of the $(5,5)$-grid. It is easy to check that this pencil has no additional base lines apart of those joining P and the points in the grid. We impose one more vanishing condition by requiring a member of this pencil to vanish at P_4. This point is distinguished by the following property: There are 10 lines passing through P_4 which meet the quadric Q_1 in two points from the grid. More precisely the points in the grid are grouped by these incidences in pairs as follows:

$$\{\{5,6\},\{7,8\},\{13,14\},\{15,16\},\{29,30\},$$
$$\{31,32\},\{33,34\},\{35,36\},\{37,38\},\{41,42\}\}.$$

There are exactly 4 more points in the configuration which have the same property, i.e., for each of them there are 10 lines intersecting Q_1 in pairs of points from the grid. These points are: P_{39}, P_{40}, P_{47} and P_{48}. Together with P_4 these points lie on the line ℓ_{24}. It is easy to check by computer that the member of the pencil vanishing at P_4 vanishes at all other points of the configuration contained in the line ℓ_{24}.

Thus in this part of the proof we established the existence of a cone C_5 of degree 5 with vertex at P vanishing at half of the points of the configuration:

$$Z_1:\ 1,4,5,6,7,8,13,14,15,16,29,30,\ldots,42,47,48,51,52,57,58.$$

This cone is smooth away of its vertex, hence there exists a smooth curve Γ_5 vanishing at all points of $\pi_P(Z_1)$. These 30 points are by construction contained in 6 disjoint lines. One can take, for example, lines from the L set and line ℓ_{24}, so that we have

$$Z_1 \subset \ell_1 \cup \ell_{24} \cup \ell_{25} \cup \ell_{32} \cup \ell_{37} \cup \ell_{44}.$$

This implies that $\pi(Z_1)$ is a $(5,6)$-complete intersection. In particular, Z_1 has the geproci property and it is a half-grid!

It is natural to wonder what properties does the residual set

$$Z_2 = Z \setminus Z_1$$

enjoy. In a sense it is surprising that exactly the same as Z_1! We have a grid determined by lines

$$L_1' = \ell_7,\ L_2' = \ell_{51},\ L_3' = \ell_{60},\ L_4' = \ell_{65},\ L_5' = \ell_{70},$$
$$M_1' = \ell_8,\ M_2' = \ell_{54},\ M_3' = \ell_{58},\ M_4' = \ell_{63},\ M_5' = \ell_{71},$$

which is contained in the quadric

$$Q_2 : x^2 + 2xy + \varphi z^2 - (\varphi - 1)\, w^2.$$

The line external to the quadric is ℓ_{17}. As before, the projection of Z_2 to $\mathbb{P}^2$ is the intersection of a smooth curve Γ'_5 of degree 5 and projection of 6 lines, for example the lines $\ell_7, \ell_{17}, \ell_{51}, \ell_{60}, \ell_{65}, \ell_{70}$. So Z_2 is also a geproci set and a half-grid.

The upshot of these considerations is that Z is contained in a cone of degree 10 with vertex at P, namely the union $C_5 \cup C'_5$ and consequently $\pi_P(Z)$ is a $(6, 10)$-complete intersection.

It remains to check that H_4 is not a half-grid. It is so for the simple reason that there are no lines containing 6 or more points from Z. $\square$

Remark 5.8

It is worth pointing out that even if Z is not a half-grid, it is a union of two such sets. In particular, it can be completely covered by a union of 12 skew 5-reach lines.
Interestingly, there are 84 different ways to cover Z by 12 disjoint 5-reach lines. These coverings are indicated in Table 5.4, where this time lines are represented by numbers only.

Appendix

Programme to calculate Waldschmidt constant for configuration H_4

```
LIB "primdec.lib";
option(redSB);
ring R2=(32003,u),(x,y,z,w),dp;
minpoly=u2-u-1;

//Ideal of point
proc point_ideal(list point) {
  matrix m[2][4] = point[1], point[2], point[3], point[4], x, y, z, w;
  ideal I = minor(m,2);
  I = std(I);
  return(I);
}

//Intersections with ideals of points
proc intersections_of_points_ideals(list points, int power){
  ideal I = point_ideal(points[1])^power;
  for(int point_index = 2; point_index <= size(points); point_index++){
    I = intersect(I, point_ideal(points[point_index])^power);
  }
  return(I);
}

//Calculate how many polys for each degree of intersections
proc how_many_polys_for_each_degree_of_intersections(ideal I){
```

```
    print_and_write("Repetitions times for degree of intersections polys:");
    int degree_repetitions = 1;
    int poly_degree = deg(I[1]);
    int current_degree = 0;

    for(int poly_idx = 2; poly_idx <= size(I); poly_idx++) {
        current_degree = deg(I[poly_idx]);
        // degree is the same, only add to repetitions
        if (current_degree == poly_degree){
            degree_repetitions = degree_repetitions + 1;

        }
        // degree is different
        else {
            print_and_write("Poly degree: " + string(poly_degree) + " repeated "
                + string(degree_repetitions) + " times");
            degree_repetitions = 1;
            poly_degree = current_degree;
        }
    }
    print_and_write("Poly degree: " + string(poly_degree) + " repeated " +
        string(degree_repetitions) + " times");
}

//Initial degree of intersections
proc alpha(ideal I){
    return(deg(I[1]));
}

//Waldschmidt constant
proc calculate_waldschmidt_constant(ideal I, number power){
    return(alpha(I)/power);
}

//Save to file and print to console
proc print_and_write(string text){
    print(text);
```

```
   write(file_to_write, text);
}

//Calculate Waldschmidt constants for next powers
proc waldschmidt_constants_for_powers(list points, int max_power){
   int initial_degree;
   number waldschmidt_constant;
   ideal I;
   for(int power = 1; power <= max_power; power++){
      print_and_write("Waldschmidt Constant for power: " + string(power));
      I = intersections_of_points_ideals(points, power);
      print_and_write("");
      print_and_write("First poly of intersections of ideals of points for
         power: " + string(power));
      print_and_write(string(I[1]));
      print_and_write("");
      how_many_polys_for_each_degree_of_intersections(I);
      initial_degree = alpha(I);
      waldschmidt_constant = calculate_waldschmidt_constant(I, power);
      print_and_write("");
      print_and_write("Initial degree for first poly: " +
         string(initial_degree));
      print_and_write("Factorize for first poly: ");
      factorize(I[1]);
      write(file_to_write, factorize(I[1]));
      print_and_write("Waldschmidt Constant: " +
         string(waldschmidt_constant));
      print_and_write("------------------------------------");
   }
}

//Save results to this file
string file_to_write = "waldschmidt_constant_for_powers_result.txt";

//List of points for configuration H4
list points =
```

```
    list(1,0,0,0), list(0,1,0,0), list(0,0,1,0), list(0,0,0,1), list(1,1,1,1),
        list(1,1,1,-1), list(1,1,-1,1), list(1,1,-1,-1), list(1,-1,1,1),
    list(1,-1,1,-1), list(1,-1,-1,1), list(1,-1,-1,-1),list(0,u,u^2,1),
        list(0,u,u^2,-1), list(0,u,-u^2,1), list(0,u,-u^2,-1),
        list(0,u^2,1,u), list(0,u^2,1,-u),
    list(0,u^2,-1,u), list(0,u^2,-1,-u), list(0,1,u,u^2), list(0,1,u,-u^2),
        list(0,1,-u,u^2), list(0,1,-u,-u^2), list(u,0,1,u^2),
        list(u,0,1,-u^2), list(u,0,-1,u^2),
    list(u,0,-1,-u^2), list(u^2,0,u,1), list(u^2,0,u,-1), list(u^2,0,-u,1),
        list(u^2,0,-u,-1), list(1,0,u^2,u), list(1,0,u^2,-u),
        list(1,0,-u^2,u), list(1,0,-u^2,-u),
    list(u,u^2,0,1), list(u,u^2,0,-1), list(u,-u^2,0,1), list(u,-u^2,0,-1),
        list(u^2,1,0,u), list(u^2,1,0,-u), list(u^2,-1,0,u),
        list(u^2,-1,0,-u), list(1,u,0,u^2),
    list(1,u,0,-u^2), list(1,-u,0,u^2), list(1,-u,0,-u^2), list(u,1,u^2,0),
        list(u,1,-u^2,0), list(u,-1,u^2,0), list(u,-1,-u^2,0),
        list(u^2,u,1,0), list(u^2,u,-1,0),
    list(u^2,-u,1,0), list(u^2,-u,-1,0), list(1,u^2,u,0), list(1,u^2,-u,0),
        list(1,-u^2,u,0), list(1,-u^2,-u,0);
int max_power = 2;

//Write to file
write(":w " + file_to_write, "");
waldschmidt_constants_for_powers(points, max_power);
```

Incidences of points and 15-reach planes

In this part we present the incidences between points in the configuration H_4 and their dual planes.

Table 5.2: Incidences of points and 15-reach planes

$V_1 : 2, 3, 4, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24$

$V_2 : 1, 3, 4, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36$

$V_3 : 1, 2, 4, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48$

$V_4 : 1, 2, 3, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60$

$V_5 : 8, 10, 11, 15, 20, 22, 26, 32, 35, 39, 44, 46, 50, 56, 59$

$V_6 : 7, 9, 12, 16, 19, 21, 25, 31, 36, 40, 43, 45, 50, 56, 59$

$V_7 : 6, 9, 12, 13, 18, 24, 28, 30, 33, 39, 44, 46, 49, 55, 60$

$V_8 : 5, 10, 11, 14, 17, 23, 27, 29, 34, 40, 43, 45, 49, 55, 60$

$V_9 : 6, 7, 12, 14, 17, 23, 26, 32, 35, 37, 42, 48, 52, 54, 57$

$V_{10} : 5, 8, 11, 13, 18, 24, 25, 31, 36, 38, 41, 47, 52, 54, 57$

$V_{11} : 5, 8, 10, 16, 19, 21, 28, 30, 33, 37, 42, 48, 51, 53, 58$

$V_{12} : 6, 7, 9, 15, 20, 22, 27, 29, 34, 38, 41, 47, 51, 53, 58$

$V_{13} : 1, 7, 10, 20, 23, 26, 27, 42, 43, 46, 47, 54, 55, 58, 59$

$V_{14} : 1, 8, 9, 19, 24, 25, 28, 41, 44, 45, 48, 54, 55, 58, 59$

$V_{15} : 1, 5, 12, 18, 21, 25, 28, 42, 43, 46, 47, 53, 56, 57, 60$

$V_{16} : 1, 6, 11, 17, 22, 26, 27, 41, 44, 45, 48, 53, 56, 57, 60$

$V_{17} : 1, 8, 9, 16, 22, 30, 31, 34, 35, 42, 43, 46, 47, 50, 51$

$V_{18} : 1, 7, 10, 15, 21, 29, 32, 33, 36, 41, 44, 45, 48, 50, 51$

$V_{19} : 1, 6, 11, 14, 24, 29, 32, 33, 36, 42, 43, 46, 47, 49, 52$

$V_{20} : 1, 5, 12, 13, 23, 30, 31, 34, 35, 41, 44, 45, 48, 49, 52$

$V_{21} : 1, 6, 11, 15, 18, 30, 31, 34, 35, 38, 39, 54, 55, 58, 59$

$V_{22} : 1, 5, 12, 16, 17, 29, 32, 33, 36, 37, 40, 54, 55, 58, 59$

$V_{23} : 1, 8, 9, 13, 20, 29, 32, 33, 36, 38, 39, 53, 56, 57, 60$

$V_{24} : 1, 7, 10, 14, 19, 30, 31, 34, 35, 37, 40, 53, 56, 57, 60$

$V_{25} : 2, 6, 10, 14, 15, 32, 34, 38, 40, 42, 44, 50, 52, 58, 60$

$V_{26} : 2, 5, 9, 13, 16, 31, 33, 37, 39, 41, 43, 50, 52, 58, 60$

$V_{27} : 2, 8, 12, 13, 16, 30, 36, 38, 40, 42, 44, 49, 51, 57, 59$

$V_{28} : 2, 7, 11, 14, 15, 29, 35, 37, 39, 41, 43, 49, 51, 57, 59$

$V_{29} : 2, 8, 12, 18, 19, 22, 23, 28, 35, 46, 48, 50, 52, 58, 60$

$V_{30} : 2, 7, 11, 17, 20, 21, 24, 27, 36, 45, 47, 50, 52, 58, 60$

$V_{31} : 2, 6, 10, 17, 20, 21, 24, 26, 33, 46, 48, 49, 51, 57, 59$

$V_{32} : 2, 5, 9, 18, 19, 22, 23, 25, 34, 45, 47, 49, 51, 57, 59$

$V_{33} : 2, 7, 11, 18, 19, 22, 23, 26, 31, 38, 40, 42, 44, 54, 56$

$V_{34} : 2, 8, 12, 17, 20, 21, 24, 25, 32, 37, 39, 41, 43, 54, 56$

$V_{35} : 2, 5, 9, 17, 20, 21, 24, 28, 29, 38, 40, 42, 44, 53, 55$

$V_{36} : 2, 6, 10, 18, 19, 22, 23, 27, 30, 37, 39, 41, 43, 53, 55$

$V_{37} : 3, 9, 11, 22, 24, 26, 28, 34, 36, 44, 47, 51, 52, 55, 56$

$V_{38} : 3, 10, 12, 21, 23, 25, 27, 33, 35, 43, 48, 51, 52, 55, 56$

$V_{39}: 3, 5, 7, 21, 23, 26, 28, 34, 36, 42, 45, 49, 50, 53, 54$

$V_{40}: 3, 6, 8, 22, 24, 25, 27, 33, 35, 41, 46, 49, 50, 53, 54$

$V_{41}: 3, 10, 12, 14, 16, 18, 20, 26, 28, 34, 36, 40, 46, 59, 60$

$V_{42}: 3, 9, 11, 13, 15, 17, 19, 25, 27, 33, 35, 39, 45, 59, 60$

$V_{43}: 3, 6, 8, 13, 15, 17, 19, 26, 28, 34, 36, 38, 48, 57, 58$

$V_{44}: 3, 5, 7, 14, 16, 18, 20, 25, 27, 33, 35, 37, 47, 57, 58$

$V_{45}: 3, 6, 8, 14, 16, 18, 20, 30, 32, 39, 42, 51, 52, 55, 56$

$V_{46}: 3, 5, 7, 13, 15, 17, 19, 29, 31, 40, 41, 51, 52, 55, 56$

$V_{47}: 3, 10, 12, 13, 15, 17, 19, 30, 32, 37, 44, 49, 50, 53, 54$

$V_{48}: 3, 9, 11, 14, 16, 18, 20, 29, 31, 38, 43, 49, 50, 53, 54$

$V_{49}: 4, 7, 8, 19, 20, 27, 28, 31, 32, 39, 40, 47, 48, 56, 58$

$V_{50}: 4, 5, 6, 17, 18, 25, 26, 29, 30, 39, 40, 47, 48, 55, 57$

$V_{51}: 4, 11, 12, 17, 18, 27, 28, 31, 32, 37, 38, 45, 46, 54, 60$

$V_{52}: 4, 9, 10, 19, 20, 25, 26, 29, 30, 37, 38, 45, 46, 53, 59$

$V_{53}: 4, 11, 12, 15, 16, 23, 24, 35, 36, 39, 40, 47, 48, 52, 59$

$V_{54}: 4, 9, 10, 13, 14, 21, 22, 33, 34, 39, 40, 47, 48, 51, 60$

$V_{55}: 4, 7, 8, 13, 14, 21, 22, 35, 36, 37, 38, 45, 46, 50, 57$

$V_{56}: 4, 5, 6, 15, 16, 23, 24, 33, 34, 37, 38, 45, 46, 49, 58$

$V_{57}: 4, 9, 10, 15, 16, 23, 24, 27, 28, 31, 32, 43, 44, 50, 55$

$V_{58}: 4, 11, 12, 13, 14, 21, 22, 25, 26, 29, 30, 43, 44, 49, 56$

$V_{59}: 4, 5, 6, 13, 14, 21, 22, 27, 28, 31, 32, 41, 42, 52, 53$

$V_{60}: 4, 7, 8, 15, 16, 23, 24, 25, 26, 29, 30, 41, 42, 51, 54$

Incidences between the H_4 configuration points and the 5-reach lines

We present the incidences between the configuration points and the 5-reach lines

Table 5.3: Incidences of points and 5-reach line

$\ell_1: 1, 29, 32, 33, 36$	$\ell_2: 1, 30, 31, 34, 35$	$\ell_3: 1, 41, 44, 45, 48$
$\ell_4: 1, 42, 43, 46, 47$	$\ell_5: 1, 53, 56, 57, 60$	$\ell_6: 1, 54, 55, 58, 59$
$\ell_7: 2, 17, 20, 21, 24$	$\ell_8: 2, 18, 19, 22, 23$	$\ell_9: 2, 37, 39, 41, 43$
$\ell_{10}: 2, 38, 40, 42, 44$	$\ell_{11}: 2, 49, 51, 57, 59$	$\ell_{12}: 2, 50, 52, 58, 60$
$\ell_{13}: 3, 13, 15, 17, 19$	$\ell_{14}: 3, 14, 16, 18, 20$	$\ell_{15}: 3, 25, 27, 33, 35$

$\ell_{16} : 3, 26, 28, 34, 36$ $\ell_{17} : 3, 49, 50, 53, 54$ $\ell_{18} : 3, 51, 52, 55, 56$
$\ell_{19} : 4, 13, 14, 21, 22$ $\ell_{20} : 4, 15, 16, 23, 24$ $\ell_{21} : 4, 25, 26, 29, 30$
$\ell_{22} : 4, 27, 28, 31, 32$ $\ell_{23} : 4, 37, 38, 45, 46$ $\ell_{24} : 4, 39, 40, 47, 48$
$\ell_{25} : 5, 13, 31, 41, 52$ $\ell_{26} : 5, 16, 33, 37, 58$ $\ell_{27} : 5, 17, 29, 40, 55$
$\ell_{28} : 5, 18, 25, 47, 57$ $\ell_{29} : 5, 21, 28, 42, 53$ $\ell_{30} : 5, 23, 34, 45, 49$
$\ell_{31} : 6, 14, 32, 42, 52$ $\ell_{32} : 6, 15, 34, 38, 58$ $\ell_{33} : 6, 17, 26, 48, 57$
$\ell_{34} : 6, 18, 30, 39, 55$ $\ell_{35} : 6, 22, 27, 41, 53$ $\ell_{36} : 6, 24, 33, 46, 49$
$\ell_{37} : 7, 14, 35, 37, 57$ $\ell_{38} : 7, 15, 29, 41, 51$ $\ell_{39} : 7, 19, 31, 40, 56$
$\ell_{40} : 7, 20, 27, 47, 58$ $\ell_{41} : 7, 21, 36, 45, 50$ $\ell_{42} : 7, 23, 26, 42, 54$
$\ell_{43} : 8, 13, 36, 38, 57$ $\ell_{44} : 8, 16, 30, 42, 51$ $\ell_{45} : 8, 19, 28, 48, 58$
$\ell_{46} : 8, 20, 32, 39, 56$ $\ell_{47} : 8, 22, 35, 46, 50$ $\ell_{48} : 8, 24, 25, 41, 54$
$\ell_{49} : 9, 13, 33, 39, 60$ $\ell_{50} : 9, 16, 31, 43, 50$ $\ell_{51} : 9, 19, 25, 45, 59$
$\ell_{52} : 9, 20, 29, 38, 53$ $\ell_{53} : 9, 22, 34, 47, 51$ $\ell_{54} : 9, 24, 28, 44, 55$
$\ell_{55} : 10, 14, 34, 40, 60$ $\ell_{56} : 10, 15, 32, 44, 50$ $\ell_{57} : 10, 19, 30, 37, 53$
$\ell_{58} : 10, 20, 26, 46, 59$ $\ell_{59} : 10, 21, 33, 48, 51$ $\ell_{60} : 10, 23, 27, 43, 55$
$\ell_{61} : 11, 14, 29, 43, 49$ $\ell_{62} : 11, 15, 35, 39, 59$ $\ell_{63} : 11, 17, 27, 45, 60$
$\ell_{64} : 11, 18, 31, 38, 54$ $\ell_{65} : 11, 22, 26, 44, 56$ $\ell_{66} : 11, 24, 36, 47, 52$
$\ell_{67} : 12, 13, 30, 44, 49$ $\ell_{68} : 12, 16, 36, 40, 59$ $\ell_{69} : 12, 17, 32, 37, 54$
$\ell_{70} : 12, 18, 28, 46, 60$ $\ell_{71} : 12, 21, 25, 43, 56$ $\ell_{72} : 12, 23, 35, 48, 52$

Sets of disjoint lines whose unions contain every point of the H_4 configuration

Table 5.4: Sets of disjoint lines whose unions contain every point of the H_4 configuration

1, 7, 17, 24, 25, 32, 37, 44, 51, 60, 65, 70	1, 7, 18, 23, 28, 35, 42, 45, 50, 55, 62, 67
1, 8, 17, 23, 25, 33, 40, 44, 54, 55, 62, 71	1, 8, 17, 24, 25, 32, 37, 44, 54, 58, 63, 71
1, 8, 18, 23, 29, 33, 40, 48, 50, 55, 62, 67	1, 8, 18, 24, 29, 32, 37, 48, 50, 58, 63, 67
1, 9, 17, 19, 28, 32, 39, 44, 54, 58, 63, 72	1, 9, 18, 20, 29, 33, 40, 47, 51, 55, 64, 67
1, 10, 17, 20, 25, 34, 37, 45, 53, 58, 63, 71	1, 10, 18, 19, 30, 33, 40, 48, 50, 57, 62, 70
1, 11, 13, 23, 29, 34, 40, 48, 50, 55, 65, 72	1, 11, 14, 24, 25, 32, 42, 47, 54, 57, 63, 71
1, 12, 13, 24, 30, 35, 37, 44, 54, 58, 64, 71	1, 12, 14, 23, 29, 33, 39, 48, 53, 60, 62, 67
2, 7, 17, 23, 28, 31, 38, 45, 49, 60, 65, 68	2, 7, 17, 24, 26, 31, 38, 43, 51, 60, 65, 70
2, 7, 18, 23, 28, 35, 42, 45, 49, 56, 61, 68	2, 7, 18, 24, 26, 35, 42, 43, 51, 56, 61, 70

2, 8, 17, 24, 26, 31, 38, 43, 54, 58, 63, 71	2, 8, 18, 23, 29, 33, 40, 48, 49, 56, 61, 68
2, 9, 17, 20, 27, 31, 40, 43, 51, 59, 65, 70	2, 9, 18, 19, 28, 36, 42, 45, 52, 56, 63, 68
2, 10, 17, 19, 26, 33, 38, 46, 51, 60, 66, 70	2, 10, 18, 20, 28, 35, 41, 45, 49, 58, 61, 69
2, 11, 13, 24, 26, 31, 41, 48, 52, 60, 65, 70	2, 11, 14, 23, 27, 35, 42, 45, 49, 56, 66, 71
2, 12, 13, 23, 28, 35, 42, 46, 54, 59, 61, 68	2, 12, 14, 24, 29, 36, 38, 43, 51, 60, 65, 69
3, 7, 17, 21, 26, 31, 39, 43, 53, 60, 62, 70	3, 7, 18, 22, 28, 32, 42, 47, 49, 57, 61, 68
3, 8, 17, 22, 27, 32, 37, 44, 49, 58, 66, 71	3, 8, 18, 21, 29, 36, 40, 43, 50, 55, 62, 69
3, 11, 13, 22, 26, 34, 42, 47, 52, 55, 66, 71	3, 11, 14, 21, 29, 32, 39, 47, 49, 60, 66, 69
3, 11, 14, 22, 27, 32, 42, 47, 49, 57, 66, 71	3, 11, 15, 19, 27, 32, 42, 46, 50, 57, 66, 70
3, 11, 16, 20, 27, 31, 40, 47, 49, 57, 64, 71	3, 12, 13, 21, 29, 36, 37, 46, 53, 60, 64, 68
3, 12, 14, 21, 29, 36, 39, 43, 53, 60, 62, 69	3, 12, 14, 22, 27, 36, 42, 43, 53, 57, 62, 71
3, 12, 15, 20, 29, 34, 39, 43, 53, 58, 61, 69	3, 12, 16, 19, 28, 36, 39, 44, 52, 60, 62, 69
4, 7, 17, 22, 26, 34, 38, 43, 51, 55, 65, 72	4, 7, 18, 21, 30, 35, 37, 45, 49, 56, 64, 68
4, 8, 17, 21, 25, 32, 37, 46, 54, 59, 63, 68	4, 8, 18, 22, 26, 33, 41, 48, 52, 55, 62, 67
4, 11, 13, 21, 26, 35, 41, 46, 54, 55, 64, 72	4, 11, 13, 22, 26, 34, 41, 48, 52, 55, 65, 72
4, 11, 14, 22, 27, 32, 41, 48, 49, 57, 65, 72	4, 11, 15, 20, 25, 34, 41, 45, 52, 55, 65, 69
4, 11, 16, 19, 26, 34, 39, 48, 52, 56, 63, 72	4, 12, 13, 21, 30, 35, 37, 46, 54, 59, 64, 68
4, 12, 13, 22, 30, 34, 37, 48, 52, 59, 65, 68	4, 12, 14, 21, 30, 35, 39, 43, 54, 59, 62, 69
4, 12, 15, 19, 30, 33, 38, 46, 54, 57, 64, 68	4, 12, 16, 20, 27, 35, 37, 46, 51, 59, 64, 67
5, 7, 15, 23, 25, 34, 42, 45, 53, 56, 61, 68	5, 7, 16, 24, 26, 31, 38, 47, 51, 60, 64, 67
5, 8, 15, 24, 25, 32, 41, 44, 54, 58, 61, 69	5, 8, 16, 23, 27, 31, 40, 48, 50, 59, 62, 67
5, 9, 13, 21, 30, 31, 40, 47, 54, 59, 64, 68	5, 9, 14, 22, 27, 32, 42, 47, 51, 59, 66, 67
5, 9, 15, 20, 27, 31, 41, 45, 53, 58, 64, 67	5, 9, 16, 19, 27, 36, 40, 44, 51, 56, 64, 72
5, 9, 16, 20, 27, 31, 40, 47, 51, 59, 64, 67	5, 10, 13, 22, 26, 34, 41, 48, 53, 58, 61, 72
5, 10, 14, 21, 25, 36, 41, 45, 53, 60, 62, 69	5, 10, 15, 19, 30, 34, 38, 45, 50, 58, 66, 69
5, 10, 15, 20, 25, 34, 41, 45, 53, 58, 61, 69	5, 10, 16, 20, 25, 34, 40, 47, 51, 59, 61, 69
6, 7, 15, 24, 30, 31, 38, 43, 50, 57, 65, 70	6, 7, 16, 23, 28, 35, 39, 44, 49, 56, 61, 72
6, 8, 15, 23, 29, 33, 38, 46, 50, 55, 66, 67	6, 8, 16, 24, 25, 36, 37, 44, 52, 56, 63, 71
6, 9, 13, 22, 28, 36, 41, 44, 52, 55, 65, 72	6, 9, 14, 21, 29, 36, 39, 43, 53, 56, 63, 72
6, 9, 15, 19, 30, 33, 39, 44, 52, 56, 66, 70	6, 9, 16, 19, 28, 36, 39, 44, 52, 56, 63, 72
6, 9, 16, 20, 28, 31, 39, 47, 52, 59, 63, 67	6, 10, 13, 21, 30, 35, 37, 46, 50, 59, 66, 70
6, 10, 14, 22, 30, 33, 38, 47, 49, 57, 66, 71	6, 10, 15, 19, 30, 33, 38, 46, 50, 57, 66, 70
6, 10, 15, 20, 25, 33, 41, 46, 53, 57, 61, 70	6, 10, 16, 19, 28, 36, 38, 46, 50, 57, 63, 72

Bibliography

[1] C. Bao, C. Freidman-Gerlicz, G. Gordon, P. McGrath, and J. Vega. Matroid automorphisms of the root system H_4. In *Proceedings of the Forty-Second Southeastern International Conference on Combinatorics, Graph Theory and Computing*, volume 207, pages 141–160, 2011.

[2] T. Bauer, S. Di Rocco, B. Harbourne, J. Huizenga, A. Lundman, P. Pokora, and T. Szemberg. Bounded negativity and arrangements of lines. *Int. Math. Res. Not. IMRN*, (19):9456–9471, 2015.

[3] T. Bauer, S. Di Rocco, B. Harbourne, J. Huizenga, A. Seceleanu, and T. Szemberg. Negative curves on symmetric blowups of the projective plane, resurgences, and Waldschmidt constants. *Int. Math. Res. Not. IMRN*, (24):7459–7514, 2019.

[4] T. Bauer and T. Szemberg. The effect of points fattening in dimension three. In *Recent advances in algebraic geometry*, volume 417 of *London Math. Soc. Lecture Note Ser.*, pages 1–12. Cambridge Univ. Press, Cambridge, 2015.

[5] C. Bocci and B. Harbourne. Comparing powers and symbolic powers of ideals. *J. Algebraic Geom.*, 19(3):399–417, 2010.

[6] C. Bocci and B. Harbourne. The resurgence of ideals of points and the containment problem. *Proc. Amer. Math. Soc.*, 138(4):1175–1190, 2010.

[7] M. V. Catalisano, E. Guardo, and Y.-S. Shin. The waldschmidt constant of special k-configurations in pn. *Journal of Pure and Applied Algebra*, 224(10):106341, 2020.

[8] L. Chiantini and J. Migliore. Sets of points which project to complete intersections, and unexpected cones. *Trans. Amer. Math. Soc.*, 374(4):2581–2607, 2021. With an appendix by A. Bernardi, L. Chiantini, G. Dedham, G. Favacchio, B. Harbourne, J.Migliore, T. Szemberg and J. Szpond.

[9] A. M. Cohen. Finite complex reflection groups. *Ann. Sci. École Norm. Sup. (4)*, 9(3):379–436, 1976.

[10] D. A. Cox, J. Little, and D. O'Shea. *Using algebraic geometry*, volume 185 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2005.

[11] W. Decker, G.-M. Greuel, G. Pfister, and H. Schönemann. SINGULAR 4-2-1 — A computer algebra system for polynomial computations. http://www.singular.uni-kl.de, 2021.

[12] R. Di Gennaro, G. Ilardi, R. M. Miró-Roig, T. Szemberg, and J. Szpond. Companion varieties for root systems and Fermat arrangements. *J. Pure Appl. Algebra*, 226(9):Paper No. 107055, 22, 2022.

[13] M. Dumnicki, B. Harbourne, U. Nagel, A. Seceleanu, T. Szemberg, and H. Tutaj-Gasińska. Resurgences for ideals of special point configurations in $\mathbf{P}^N$ coming from hyperplane arrangements. *J. Algebra*, 443:383–394, 2015.

[14] L. Ein, R. Lazarsfeld, and K. E. Smith. Uniform bounds and symbolic powers on smooth varieties. *Invent. Math.*, 144(2):241–252, 2001.

[15] D. Eisenbud. *Commutative algebra*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. With a view toward algebraic geometry.

[16] M. Fekete. Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten. *Math. Z.*, 17(1):228–249, 1923.

[17] T. Fujita. Approximating Zariski decomposition of big line bundles. *Kodai Math. J.*, 17(1):1–3, 1994.

[18] W. Fulton. *Intersection theory*, volume 2 of *Ergeb. Math. Grenzgeb., 3. Folge*. Springer, Cham, 1984.

[19] W. Fulton. *Algebraic curves*. Advanced Book Classics. Addison-Wesley Publishing Company, Advanced Book Program, Redwood City, CA, 1989. An introduction to algebraic geometry, Notes written with the collaboration of Richard Weiss, Reprint of 1969 original.

[20] A. V. Geramita, B. Harbourne, and J. Migliore. Star configurations in $\mathbb{P}^n$. *J. Algebra*, 376:279–299, 2013.

[21] C. G. Gibson. *Elementary Geometry of Algebraic Curves: An Undergraduate Introduction*. Cambridge University Press, 1998.

[22] B. Harbourne. Asymptotics of linear systems, with connections to line arrangements. In *Phenomenological approach to algebraic geometry*, volume 116 of *Banach Center Publ.*, pages 87–135. Polish Acad. Sci. Inst. Math., Warsaw, 2018.

[23] B. Harbourne, J. Migliore, U. Nagel, and Z. Teitler. Unexpected hypersurfaces and where to find them, arXiv:1805.10626, accepted for publication in Michigan Math. J.

[24] R. Kane. *Reflection groups and invariant theory*, volume 5 of *CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC*. Springer-Verlag, New York, 2001.

[25] M. Nagata. On the 14-th problem of Hilbert. *Amer. J. Math.*, 81:766–772, 1959.

[26] P. Pokora, T. Szemberg, and J. Szpond. Unexpected properties of the Klein configuration of 60 points in $\mathbb{P}^3$. *Research in Pairs 2020, OWP-2020-19*, 2020, to appear in Michigan Math. J.

[27] G. C. Shephard and J. A. Todd. Finite unitary reflection groups. *Canadian J. Math.*, 6:274–304, 1954.

[28] J. Szpond. A few introductory remarks on line arrangements. In *Analytic and algebraic geometry. 3*, pages 201–212. Wydawn. Uniw. Łódzkiego, Łódź, 2019.

[29] P. Wiśniewska and M. Zięba. Generic projections of the H_4 configuration of points. *Adv. in Appl. Math.*, 142:Paper No. 102432, 9, 2023.