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FIXED POINT THEOREMS IN ORDERED SPACES

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TWIERDZENIA O PUNKTACH STAŁYCH W PRZESTRZENIACH UPORZĄDKOWANYCH

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Streszczenie

Niniejsza rozprawa bada warunki wystarczające na istnienie punktów stałych dla odwzorowań monotonicznie G-nieoddalających i wielowartościowych odwzorowań monotonicznie G-nieoddalających w przestrzeniach metrycznych oraz w przestrzeniach modularnych wyposażonych w strukturę digrafu. W szczególnosci bada odwzorowania monotoniczne i wielowartościowe odwzorowania monotoniczne w uporządkowanych przestrzeniach metrycznych i modularnych.

W tym celu podchodzimy do teorii punktu stałego w dwóch kierunkach. Pierwszy kierunek związany jest z miarą niezwartości. Korzystając z założeń Darbo i Sadovskiego, dowodzimy istnienia punktów stałych odwzorowań monotonicznych i wielo-wartościowych odwzorowań monotonicznych. Przedstawiamy także różne zastosowania uzyskanych wyników do kilku modeli: równań całkowych typu Hammersteina, równań całkowych typu Volterry, problemów początkowych pierwszego rzędu z nieciągłościami oraz funkcyjnych inkluzji całkowych.

W drugim kierunku rozważamy struktury geometryczne w przestrzeniach metrycznych i modularnych. W rezultacie powstało wiele własności, które zostały użyte, żeby pokazać istnienie punktów stałych odwzorowań monotonicznie G-nieoddalających oraz wielowartościowych odwzorowań monotonicznie G-nieoddalających.

Abstract

This dissertation investigates the sufficient conditions for the existence of fixed points for monotone G-nonexpansive mappings and monotone G-nonexpansive multivalued mappings in metric spaces as well as modular spaces equipped with a digraph. In particular, it examines monotone mappings and monotone multivalued mappings in metric and modular ordered spaces.

For these purposes, we approach fixed point theory in two directions. The first direction is related to the measure of noncompactness. Using the assumptions of Darbo and Sadovskiĭ, we prove the existence of fixed points of monotone mappings and monotone multivalued mappings. We also present various applications of the achieved results to several models: integral equations of Hammerstein type, integral equations of Volterra type, first order initial value problems with discontinuities and functional integral inclusions.

In the remaining direction, we consider geometric structures for metric spaces and modular spaces. Consequently, numerous properties have been established and used to show the existence of fixed points of monotone G-nonexpansive mappings and monotone G-nonexpansive multivalued mappings.

List of symbols

	General notations
$\mathbb{R}$	the set of all real numbers
$\mathbb{R}_+$	$\mathbb{R}_+ := [0, \infty)$
I	$I := [0, 1]$
$\mathbb{N}$	$\mathbb{N} := \{1, 2, \ldots\}$
$\operatorname{diam}(A)$	the diameter of a subset A
$\preceq_X$	a partial order on the set X
$\preceq_k$	a partial order on the set $\mathbb{R}^k$
$(\leftarrow, a], [a, \rightarrow)$	order intervals
$G = (V(G), E(G))$	a digraph G with vertices $V(G)$ and edges $E(G)$
$(\leftarrow, a]_G, [a, \rightarrow)_G,$	G-intervals along the walks
MNCs	measure of noncompactness
$(x_n)_n$	a sequence $\{x_1, x_2, ...\}$
	Spaces and families
$\mathbb{B}(X)$	the family of all nonempty bounded subsets of metric space X
$\mathbb{CL}(X)$	the family of all nonempty closed subsets of metric space X
$\mathbb{CP}(X)$	the family of all nonempty compact subsets of metric space X
$\mathcal{C}(I, \mathbb{R})$	the space of all continuous real-valued functions defined on I
$\mathcal{BC}(\mathbb{R}_+)$	the space of all bounded continuous real functions on $\mathbb{R}_+$
$\mathbb{CL}_\rho(X)$	the family of all nonempty closed subsets of modular space X_ρ
$\mathbb{CP}_\rho(X)$	the family of all nonempty compact subsets of modular space X_ρ

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Chapter 1

Introduction

In mathematics, a fixed point refers to a specific element in the domain that remains unchanged when the mapping acts on it. More formally, an element x_0 belonging to a nonempty set X is said to be a fixed point of the mapping $T : X \to X$ if $T(x_0) = x_0$.

Fixed point theory and its applications are fascinating areas of study in modern mathematics. The known results for monotone mappings play a major role in determining solutions for differential equations, integral equations, systems of nonlinear equations, functional equations, etc. (see [18], [34]). Additionally, the study of monotone multivalued mappings, a branch of mathematics that has gained significant attention in recent decades, has practical applications in fields such as convex optimization, optimal control theory and differential inclusions (see [62]). Their relevance extends beyond mathematics with applications in diverse scientific disciplines including computer science, control theory, game theory, mathematical physics, biology, economics and many others (see [88], [112]).

The Knaster-Tarski fixed point theorem (see [79]) for monotone mappings is a well-known result due to its applications in the field of denotational semantics for programming languages. It says that any monotone mapping $T : X \to X$ on a complete lattice X possesses a fixed point. Unlike Banach contraction principle, the Knaster-Tarski fixed point theorem does not provide an algorithm to approximate fixed points. If we additionally equip X with a metric, it will provide us with a metric structure rich enough to obtain new fixed point theorems. Metric spaces also allows us to use measures of noncompactness on them. Fixed point theory with an approach involving measures of noncompactness is a powerful tool for assessing the existence of solutions of integral and differential equations. Darbo [36] was the first to apply the Kuratowski measure of noncompactness α to show the existence of fixed points for any continuous mapping $T : X \to X$ provided that there exists $k \in [0, 1)$ such that

$$\alpha(T(\Omega)) \leq k\alpha(\Omega),$$

where Ω belongs to the set $\mathbb{B}(X)$ of all bounded subsets of X. Note that Darbo's proof includes the existential aspect of Banach's fixed point theorem. Many researchers have extended this result and also provided applications of various types of functional equations (see [2], [3], [4], [6], [21], [39], [75]). The main research direction in these results involves considering the extension conditions of the Darbo's assumption for continuous operators defined on closed bounded convex subsets of a Banach space. The argument used in the proofs of these results is similar to

Darbo's. Darbo's theorem has also been extended to the case of monotone continuous mappings (see [5], [96], [113]). So far, many applications of the theorems have been shown (see [4], [5], [21], [96], [113]).

Sadovskiĭ [106] generalized Darbo's result for condensing operators that are continuous and satisfy the inequality

$$\alpha(T(\Omega)) < \alpha(\Omega).$$

There are also numerous applications that illustrate Sadovskiĭ's result (see [3], [21]). It is worth emphasizing that the continuity of the map T plays an important role in the proof of Sadovskiĭ. Monotone mappings are not necessarily continuous. Hence, in extending Sadovskiĭ's theorem to monotone mappings, the continuity of mappings has been modified (see [6], [14]). In 2018, considering on partially ordered Hausdorff topological spaces with compact order intervals, Espínola and Wiśnicki [47] showed that the set of fixed point of any monotone mapping T is nonempty provided that there exists c such that $c \preceq T(c)$. In 2023, Taoudi [111] gave a generalization of Espínola-Wiśnicki's theorem. It confirms that on a partially ordered Hausdorff topological space X, if we assume that every order interval is closed and C is a nonempty closed subset of X, then any monotone mapping $T : C \to C$ has a fixed point provided that for any totally ordered subset Ω of C, $\overline{T(\Omega)}$ is a compact subset of X. Clearly, the compactness is an important assumption in these results. For any measure of noncompactness μ on a metric space X, $\overline{\Omega}$ is compact if $\mu(\Omega) = 0$, where $\Omega \in \mathbb{B}(X)$. By this relation and the above results, we give counterparts of Darbo–Sadovskiĭ's theorem for monotone mappings in metric spaces without continuity. It is also natural to extend these results to the case of monotone multivalued mappings. The study of fixed points of multivalued mappings is closely linked to the study of the solution of functional integral inclusions. Research related to the measures of noncompactness has also received much attention in recent years (see [17], [41], [43], [44], [57], [60], [97], [88], [112] [114]). In these results, it is necessary to assume that the set-valued function under consideration is either lower semi-continuous (upper semi-continuous) or continuous with respect to the Hausdorff metric $H(\cdot, \cdot)$ in its domain. Subsequently, applications of Carathéodory's condition for multi-functions are a commonly employed method when proving these existence theorems. More recently, several authors have established many results for monotone multivalued mappings. These monotone multivalued mappings satisfy some kind of continuity (see [40], [57]). Our next aim is to consider the existence of a fixed point of monotone maps $T : C \to \mathbb{CL}(C)$ without continuity, where C is a closed bounded subset of a metric space X, where $\mathbb{CL}(C)$ denotes the class of all non-empty closed subsets of X.

Recent results in fixed point theory also attempt to be extended to G-monotone mappings in spaces equipped with a digraph G. However, the approaches of Espínola-Wiśnicki and Taoudi only work on partially ordered sets, it is difficult to extend them to the case of sets endowed with a digraph. We know that the original motivation of Knaster-Tarski's theorem [79] was to find an invariant set for suitably defined monotone operator on the power set. If such sets are available, they will enable us to use iteration in next steps of the proof. This convenience also applies to G-monotone mappings. Hence, the identification of invariant sets through given G-monotone mappings is the first important step in the process of proving the existence of their fixed points. It is also the second purpose in my dissertation. For this purpose, we extend the assumption of Espínola-Wiśnicki for the family of G-intervals along walks as follows:

(**P1**) *Any family of G-intervals along walks of the form* $[a, b]_G$ *or* $[a, \to)_G$ *with the finite intersection property has a nonempty intersection.*

By combining the G-monotonicity of the mapping with Kuratowski–Zorn's lemma, we can ascertain the existence of a G-interval that remains invariant under any G-monotone map.

The above obtained results allow us to extend a lot of results for monotone nonexpansive mappings to monotone G-nonexpansive mappings. Fixed point theory for nonexpansive mappings has its origins in the papers by Browder [28], Göhde [55] and Kirk [77] that were published in 1965. The initial positive results were all obtained when the domain was a convex subset of a Banach space. Therefore, a convex structure is essential for the domain of monotone nonexpansive mappings when investigating their fixed points. The current research has primarily focused on Banach spaces (see [7], [12], [18], [71], [108], [110]), hyperbolic metric spaces (see [9], [37], [109]) and modular spaces (see [1], [8], [24], [37]) that are equipped with uniform convexity. Most of the spaces in the mentioned results have the following property:

(**P2**) *Every nonincreasing sequence of nonempty bounded closed (ρ-bounded ρ-closed for modular spaces) convex subsets of X has a nonempty intersection.*

With the use of uniform convexity (see Definition 4.1, [68]), we are going to prove the following property which is equivalent to (P2):

(**P3**) *Any family of nonempty bounded closed (ρ-bounded ρ-closed for modular spaces) convex subsets of X satisfying the finite intersection property, has a nonempty intersection.*

Therefore, property (P1) is satisfied under the assumption that G-intervals along walks are closed, bounded and convex subsets of X. This assumption is natural in the context of function spaces. With property (P3), it allows us to approach the research direction of Kirk [77] in the study of the fixed point problem for monotone G-nonexpansive mappings in various spaces. Using techniques associated with the asymptotic center and the asymptotic radius, Kirk showed that any reflexive Banach space with normal structure has the fixed point property for nonexpansive mappings. Khamsi extended this result for metric spaces ([64]) and for modular spaces ([65]). Our next purpose is to extend these results to monotone G-nonexpansive mappings in geodesic spaces.

There exist numerous convex structures of Banach spaces that imply normal structure and consequently the fixed point property for multivalued mappings such as uniform convexity, nearly uniform convexity etc. Hence, it is natural to investigate whether these structures can also imply the existence of fixed points of monotone G-nonexpansive multivalued mappings in hyperbolic metric spaces and modular spaces. Recent results are provided for monotone G-nonexpansive multivalued mappings $T : C \to \mathbb{CL}(C)$, where C is a G-compact subset in hyperbolic metric space (see [9]). We consider the case, where C is a closed (resp. ρ-closed) subset in a hyperbolic metric space (resp. modular space). Then we show the existence of fixed points for monotone G-nonexpansive multivalued mapping $T : C \to \mathbb{CP}(C)$, where $\mathbb{CP}(C)$ is the family of all compact subsets of C.

For the above purposes, the structure of the thesis is outlined as follows: Chapter 1 provides a comprehensive overview of the fixed point theory concerning monotone mappings. It analyzes findings established by previous researchers, including ones that have not been addressed in prior studies. It also presents the purpose and research methods used in the thesis. Finally, an overview of the thesis structure is provided.

In Chapter 2, we introduce fundamental concepts regarding ordered metric spaces and measures of noncompactness. Next, we establish fixed point theorems for monotone mappings using measures of noncompactness. We also prove the existence of common fixed points for a commutative family of monotone mappings. Furthermore, we extend these results to monotone multivalued mappings. As applications, we present examples regarding the existence of solutions of integral equations of Hammerstein type, integral equations of Volterra type, first order initial value problems with discontinuities and functional integral inclusions.

In the first section of Chapter 3, we provide some basic definitions in graph theory. In the next section, we show the existence of invariant G-intervals under G-monotone mappings. Following this, we present several fixed point theorems for monotone G-nonexpansive mappings and monotone G-asymptotically nonexpansive mappings in Banach spaces. We also give an application of Volterra type integral equation to illustrate our results.

Chapter 4 is dedicated to fixed point theorems concerning G-monotone mappings, monotone G-nonexpansive mappings and monotone G-nonexpansive multivalued mappings in geodesic spaces. Firstly, we introduce fundamental concepts regarding geodesic spaces, uniformly convex structures and their properties. Based on these properties, we obtain existence theorems of the aforementioned mappings.

Following the structure of Chapter 4, Chapter 5 introduces fundamental concepts of modular spaces and their uniformly convex structures. Additionally, we present essential properties of uniformly convex modular spaces required for our proofs. Subsequently, we obtain the existence of fixed points for monotone G_ρ-nonexpansive mappings and monotone G_ρ-nonexpansive multivalued mappings in these spaces.

Chapter 2

Fixed point theorems via measures of noncompactness

In this chapter, we introduce basic concepts related to ordered metric spaces and measures of noncompactness. We also provide brief illustrative examples intended for applications in the next sections. Following that, we establish several fixed point theorems for monotone mappings and monotone multivalued mappings in ordered metric spaces. Finally, these results will be used to prove the existence of solutions for integral equations, differential equations and functional integral inclusions.

2.1. Preliminaries

2.1.1 Ordered metric spaces

Let us start this section by introducing basic definitions concerning partial orders and the Kuratowski-Zorn lemma. These concepts can be found in [56].

Definition 2.1.1. Let X be a nonempty set. We say that a relation $\preceq$ is a partial order on X if for any $x_1, x_2, x_3 \in X$, the following conditions are satisfied:

(i) Reflexivity: $x_1 \preceq x_1$;

(ii) Antisymmetry: if $x_1 \preceq x_2$ and $x_2 \preceq x_1$ then $x_1 = x_2$;

(iii) Transitivity: if $x_1 \preceq x_2$ and $x_2 \preceq x_3$ then $x_1 \preceq x_3$.

A poset $(X, \preceq)$ is a nonempty set X equipped with a partial order relation $\preceq$ on it. Order intervals mean the sets of the form

$$[a, \rightarrow) := \{x \in X : a \preceq x\},$$
$$(\leftarrow, b] := \{x \in X : x \preceq b\},$$
$$[a, b] := [a, \rightarrow) \cap (\leftarrow, b]$$

for every $a, b \in X$.

Example 2.1.2. 1) It is easy to prove that any subset X of $\mathbb{R}$ forms a poset when it is equipped with the standard less than or equal relation "$\leq$". Order intervals in X are defined by $[a, \rightarrow) = [a, +\infty) \cap X$, $(\leftarrow, a] = (-\infty, a] \cap X$ for $a \in X$.

Take $k \in \mathbb{N}$. On $\mathbb{R}^k$, we define a relation as follows:

$$x \leq_k y \Leftrightarrow x_i \leq y_i \text{ for all } i \in \{1, \ldots, k\},$$

for $x = (x_1, \ldots, x_k), y = (y_1, \ldots, y_k) \in \mathbb{R}^k$. We can show that $\leq_k$ is a partial order. Then for any subset X of $\mathbb{R}^k$, $(X, \leq_k)$ is a poset, and order intervals in X are sets of the form

$$[x, \rightarrow) = \{y = (y_1, \ldots, y_k) \in X : x_i \leq y_i \text{ for all } i \in \{1, \ldots, k\}\},$$
$$(\leftarrow, x] = \{y = (y_1, \ldots, y_k) \in X : x_i \geq y_i \text{ for all } i \in \{1, \ldots, k\}\}.$$

We note that for $k = 1$ we have $\leq_1 := \leq$.

2) Assume that $k, m \in \mathbb{N}$. Let X be a nonempty subset of $\mathbb{R}^k$ and Y be a nonempty subset of the poset $(\mathbb{R}^m, \leq_m)$. Denote by $\mathcal{F}(\mathbb{R}^k, \mathbb{R}^m)$ the set of all functions f from $\mathbb{R}^k$ to $\mathbb{R}^m$. We consider a relation $\preceq_{\mathcal{F}}$ defined by

$$h \preceq_{\mathcal{F}} g \Leftrightarrow h(x) \leq_m g(x) \quad \forall x \in \mathbb{R}^k,$$

for $h, g \in \mathcal{F}(\mathbb{R}^k, \mathbb{R}^m)$. Clearly, $\preceq_{\mathcal{F}}$ is a partial order. Thus $(\mathcal{F}(\mathbb{R}^k, \mathbb{R}^m), \preceq_{\mathcal{F}})$ is a poset.

Definition 2.1.3. Let $(X, \preceq)$ be a poset and Y be a subset of X.

(i) An element $x_0 \in X$ is an upper bound of subset Y if $Y \subseteq (\leftarrow, x_0]$, i.e., $y \preceq x_0$ for every $y \in Y$.

(ii) If an upper bound x_0 of Y belongs to Y, x_0 is called the maximum of Y, and denoted as $x_0 = \max A$.

(iii) The element $y_0 \in X$ is said to be the supremum of Y, denoted $y_0 = \sup Y$ if y_0 is an upper bound of Y and $x \in [y_0, \rightarrow)$ for every upper bound x of Y.

(iv) An element $z_0 \in Y$ is called maximal of Y if there does not exist any element $z_1 \in Y$ such that $z_0 \preceq z_1$ and $z_0 \neq z_1$.

The notions of a lower bound, minimum, infimum and a minimal element of A are defined similarly.

Definition 2.1.4. Let $(X, \preceq)$ be a poset and Y be a subset of X.

(i) The set Y is said to be a chain (or totally ordered) if $x \in (\leftarrow, y]$ or $y \in (\leftarrow, x]$ for any $x, y \in Y$.

(ii) The set Y is called directed if for each pair $x, y \in Y$ there exists $z \in Y$ such that $x, y \in (\leftarrow, z]$.

Clearly, if A is a chain, A is also a directed set. The converse of this statement is not correct.

Example 2.1.5. Any subset of $\mathbb{R}$ is a chain. This statement doesn't hold for $(\mathbb{R}^k, \preceq_k)$ where $k > 1$. Indeed we take a subset $Y = \{y = (y_1, ..., y_k) : |y_i| \leq 1 \text{ for all } i \in \{1, ..., k\}\}$. Consider $z = (1/2, 0, 0, ..., 0)$ and $t = (0, 1/2, 0, ..., 0)$. Clearly, $z, t \in Y$, $z \notin (\leftarrow, t]$ and $t \notin (\leftarrow, z]$. Furthermore, we can choose $v = (1, 1, 0, ..., 0) \in Y$ such that $z, t \in (\leftarrow, v]$. Thus Y is a directed set but it is not a chain.

The following lemma, known as the Kuratowski–Zorn lemma, will be used in proving some theorems of this thesis. It provides a certain sufficient condition for the existence of a maximal element in a set endowed with a partial order.

Lemma 2.1.6 (Kuratowski-Zorn lemma). *Let $(X, \preceq)$ be a poset. Assume that every chain in X has an upper bound in X. Then the set X contains at least one maximal element.*

Definition 2.1.7. We say that a sequence $(x_n)_n$ in a poset $(X, \preceq)$ is nondecreasing (resp. nonincreasing) if $x_n \preceq x_{n+1}$ (resp. $x_{n+1} \preceq x_n$) for every $n \geq 1$. A sequence is called monotone if it is nondecreasing or nonincreasing.

In this thesis, when referring to monotone sequences, it is implicitly understood without explicit specification that we are discussing nondecreasing sequences. In this chapter, we are going to work on ordered metric spaces that have the following definition.

Definition 2.1.8 ([56]). Let (X, d) be a metric space, and $\preceq$ be a partial order on X. An ordered metric space is a triple $(X, d, \preceq)$ such that in the metric space (X, d), order intervals $[x, \rightarrow)$, $(\leftarrow, x]$ are closed sets for every $x \in X$.

Example 2.1.9. 1) The Euclidean metric d, the metric d_1, and the maximum metric d_2 are respectively defined on $\mathbb{R}^k$ $(k \in \mathbb{N})$ by

$$\begin{aligned} d(x, y) &= \Big((x_1 - y_1)^2 + \ldots + (x_k - y_k)^2\Big)^{\frac{1}{2}}. \\ d_1(x, y) &= |x_1 - y_1| + \ldots + |x_k - y_k|. \\ d_2(x, y) &= \max\{|x_1 - y_1|, \ldots, |x_k - y_k|\}. \end{aligned}$$

Then $(\mathbb{R}^k, d, \preceq_k)$, $(\mathbb{R}^k, d_1, \preceq_k)$ and $(\mathbb{R}^k, d_2, \preceq_k)$ are ordered metric spaces.

2) Put $m_0 := \{x = (x_1, x_2, x_3, \ldots) : x_n = x_{n+1} = \ldots = 0 \text{ for some } n \in \mathbb{N}\}$. We equip this space with the following metric

$$d_{m_0}(x, y) = \sup_{i \in \mathbb{N}} |x_n - y_n|$$

for all $x = (x_1, x_2, x_3, \ldots), y = (y_1, y_2, y_3, \ldots) \in m_0$. Obviously, (m_0, d_{m_0}) is a metric space. We consider a partial order on m_0 as follows:

$$x \preceq_{m_0} y \Leftrightarrow x_i \leq y_i \text{ for all } i \in \mathbb{N},$$

for $x = (x_1, x_2, x_3, \ldots), y = (y_1, y_2, y_3, \ldots) \in m_0$. Order intervals in m_0 are defined by

$$\begin{aligned} [x, \rightarrow) &= \{y = (y_1, y_2, \ldots) \in X : x_i \leq y_i \ \forall i \geq 1\}, \\ (\leftarrow, x] &= \{y = (y_1, y_2, \ldots) \in X : x_i \geq y_i \ \forall i \geq 1\}. \end{aligned}$$

It is not difficult to prove that $(m_0, d_{m_0}, \preceq_{m_0})$ is an ordered metric space.

Definition 2.1.10 ([56]). An ordered Banach space is a triple $(X, \|\cdot\|, \preceq)$ such that $(X, \|\cdot\|)$ is a Banach space, and $(X, d, \preceq)$ is an ordered metric space, where d is the metric generated by the norm $\|\cdot\|$.

Obviously, $(X, \|\cdot\|, \preceq)$ is an ordered Banach space if and only if order intervals are closed sets in the Banach space $(X, \|\cdot\|)$.

Example 2.1.11. Put $I := [0,1] \subset \mathbb{R}$. Let $\mathcal{C}(I,\mathbb{R})$ represent the space of all continuous real-valued functions defined on I. We consider the following maximum norm

$$\|f\|_c = \max_{x\in I} |f(x)|$$

for every $f \in \mathcal{C}(I,\mathbb{R})$. With this norm, $\mathcal{C}(I,\mathbb{R})$ is a Banach space. On $\mathcal{C}(I,\mathbb{R})$, we defined a relation as follows:

$$f \preceq_c g \Leftrightarrow f(x) \leq g(x) \quad \forall x \in I,$$

for every $f, g \in \mathcal{C}(I,\mathbb{R})$. It is a simple matter to show that order intervals in $\mathcal{C}(I,\mathbb{R})$ are closed sets in the Banach space $(\mathcal{C}(I,\mathbb{R}), \|\cdot\|_c)$. Hence $(\mathcal{C}(I,\mathbb{R}), \|\cdot\|_c, \preceq_c)$ is an ordered Banach space.

The next results are special properties of convergent monotone sequences in ordered metric spaces.

Lemma 2.1.12 ([56]). *Let $(x_n)_n$ be a sequence in a poset $(X, \preceq)$.*

(i) If $(x_n)_n$ is a chain, it has a monotone subsequence.

(ii) If $(x_n)_n$ is nondecreasing (resp. nonincreasing), then it has the supremum (resp. the infimum) y_0 if and only if y_0 is the supremum (resp. the infimum) of some of its subsequences.

Proposition 2.1.13 ([56]). *If a nondecreasing (resp. nonincreasing) sequence $(x_n)_n$ in an ordered metric space $(X, d, \preceq)$ has a limit point y_0, then $y_0 = \sup_n x_n$ (resp. $y_0 = \inf_n x_n$).*

Proposition 2.1.14 ([56]). *A monotone sequence in an ordered metric space X converges if its each subsequence has a limit point.*

The following result provides necessary and sufficient conditions for the convergence of monotone sequences in a chain of ordered metric spaces.

Proposition 2.1.15 ([56]). *If C is a chain in an ordered metric space, then each monotone sequence of C converges if and only if $\overline{C}$ is compact.*

2.1.2 Measure of noncompactness

Let (X, d) be a complete metric space. The symbol $\overline{B}_x(r)$ will denote the closed ball centered at $x \in X$ with radius $r > 0$. Denote by $\mathbb{B}(X)$ the family of all nonempty bounded subsets of X, $\mathbb{CL}(X)$ the family of all nonempty closed subsets of X, $\mathbb{CP}(X)$ the family of all nonempty compact subsets of X and $\operatorname{diam}(A)$ the diameter of a subset A of X.

Definition 2.1.16 (compare [113]). A measure of noncompactness (MNCs for short) defined on a complete metric space (X, d) is a function $\nu : \mathbb{B}(X) \to [0, \infty)$ such that for any $\Omega_1, \Omega_2 \in \mathbb{B}(X)$, we have

(i) If $\nu(\Omega_1) = 0$ then $\overline{\Omega}_1 \in \mathbb{CP}(X)$;

(ii) $\nu(\Omega_1) = \nu(\overline{\Omega}_1)$;

(iii) $\nu(\Omega_1 \cup \Omega_2) = \max\{\nu(\Omega_1), \nu(\Omega_2)\}$.

In Definition 2.1.16, if we replace condition i) with condition

(i') $\nu(\Omega_1) = 0$ if and only if $\overline{\Omega}_1 \in \mathbb{CP}(X)$,

then we say that ν is a regular measure of noncompactness on X.

From Definition 2.1.16, we infer the following properties:

(iv) If $\Omega_1 \subseteq \Omega_2$ then $\nu(\Omega_1) \leq \nu(\Omega_2)$,

(v) $\nu(\Omega_1 \cap \Omega_2) \leq \min\{\nu(\Omega_1), \nu(\Omega_2)\}$,

(vi) If $(\Omega_n)_n \subseteq \mathbb{CL}(X) \cap \mathbb{B}(X)$ such that $\Omega_{n+1} \subseteq \Omega_n$ for every $n \geq 1$ and $\lim\limits_{n\to\infty} \mu(\Omega_n) = 0$, then the set $\Omega_\infty = \bigcap_{i=1}^{\infty} \Omega_n$ is nonempty and compact.

(vii) If ν is a regular measure of noncompactness on X, and $\Omega = \{x_1, \ldots, x_n\} \subseteq X$ then $\nu(\Omega) = 0$.

Moreover, if X is a Banach space, the regular measure of noncompactness ν can enjoy the following properties:

(viii) $\nu(t\Omega_1) = |t|\nu(\Omega_1)$ for any number t.

(ix) $\nu(\Omega_1 + \Omega_2) \leq \nu(\Omega_1) + \nu(\Omega_2)$.

(x) $\nu(x_0 + \Omega_1) = \nu(\Omega_1)$ for any $x_0 \in X$.

(xi) $\nu(\mathrm{co}(\Omega_1)) = \nu(\Omega_1)$, where $\mathrm{co}(\Omega_1)$ is the convex hull of Ω_1.

We are going to present some measures of noncompactness. These examples will be used in Section 2.5.

Example 2.1.17 ([80]). Let (X, d) be a complete metric space. The Kuratowski MNCs is defined by

$$\alpha(\Omega) = \inf\{\varepsilon > 0 : \Omega \subseteq \bigcup_{k=1}^{m} \Omega_k, \Omega_k \subset X, \operatorname{diam} \Omega_k \leq \varepsilon \text{ for all } 1 \leq k \leq m\}$$

for every $\Omega \in \mathcal{B}(X)$.

Example 2.1.18. We consider the Banach space $(\mathcal{C}(I, \mathbb{R}), \|\cdot\|_{\mathcal{C}})$ as mentioned in Example 2.1.11. Now, we take $\Omega \in \mathbb{B}(\mathcal{C}(I, \mathbb{R}))$, $f \in \Omega$ and $\delta > 0$. Put

$$\begin{aligned}
\Psi(f, \delta) &:= \sup\{|f(x) - f(y)| : x, y \in I, |x - y| \leq \delta\}, \\
\Psi(\Omega, \delta) &:= \sup_{f \in \Omega} \Psi(f, \delta), \\
\Psi_0(\Omega) &:= \lim_{\delta \to 0} \Psi(\Omega, \delta).
\end{aligned}$$

The function Ψ_0 is a regular MNCs on $\mathcal{C}(I; \mathbb{R})$ (see [22]).

Example 2.1.19. Consider the space $\mathcal{BC}(\mathbb{R}_+)$ of all bounded continuous real-functions on $\mathbb{R}_+ := [0, \infty)$ with the standard sup-norm

$$\|f\|_{\mathcal{BC}} = \sup_{x \geq 0} |f(x)|$$

for every $f \in \mathcal{BC}(\mathbb{R}_+)$. Clearly, $(\mathcal{BC}(\mathbb{R}_+), \|\cdot\|_{\mathcal{BC}})$ is a Banach space. Take $\Omega \in \mathbb{B}(\mathcal{BC}(\mathbb{R}_+))$, $f \in \Omega$, $L > 0$ and $\varepsilon > 0$. We define

$$\begin{aligned}
\Psi^L(f, \varepsilon) &:= \sup\{|f(x) - f(y)| : x, y \in [0, L], |x - y| \leq \varepsilon\}, \\
\Psi^L(\Omega, \varepsilon) &:= \sup_{f \in \Omega} \Psi^L(f, \varepsilon), \\
\Psi_0^L(\Omega) &:= \lim_{\varepsilon \to 0} \Psi^L(\Omega, \varepsilon), \\
\Psi_0(\Omega) &:= \lim_{L \to \infty} \Psi_0^L(\Omega),
\end{aligned}$$

and consider the function

$$b(\Omega) := \lim_{L \to \infty} \left\{ \sup_{f \in \Omega} \{\sup\{|f(x) - f(y)| : x, y \geq L\}\} \right\}.$$

Put

$$\Psi_1(\Omega) := \Psi_0(\Omega) + b(\Omega).$$

In [20], Banaś showed that Ψ_1 is a MNCs on $\mathcal{BC}(\mathbb{R}_+)$.

2.2. Fixed points of monotone mappings

This section has two parts. In the first part, focusing on ordered metric spaces, we show counterparts of Darbo's theorem for a monotone, not necessarily continuous, mapping $F : X \to X$ under the assumption of Darbo using measures of noncompactness. In the next part, we prove the existence of common fixed points for a commutative family of monotone mappings under the assumption of Sadovskiĭ [106] using regular measures of noncompactness. Our existence results exploit the arguments of Darbo [36], Sadovskiĭ [106], and Taoudi [111]. Firstly, we recall the definition of monotone mappings.

2.2.1 Fixed points of monotone mappings

Definition 2.2.1 ([56]). Let $(X, \preceq)$ be a poset. A mapping $T : X \to X$ is called monotone (or order-preserving) if for all $x, y \in X$ such that $x \preceq y$, it satisfies $T(x) \preceq T(y)$.

A point $x \in X$ is called a fixed point of T if $T(x) = x$. The set of fixed points of T will be denoted by $\mathrm{Fix}(T)$.

Example 2.2.2. 1) Assume that $k \geq 1$. On the poset $(\mathbb{R}^k, \leq_k)$, the mapping T is defined by

$$T(x) = (x_1 + 1, x_2^3 + 3, ..., x_k^{2k+1} + (2k + 1)),$$

for $x = (x_1, ..., x_k) \in \mathbb{R}^k$. Since $f(t) = t^{2n+1} + (2n + 1)$ is nondecreasing on $\mathbb{R}$ for every $n \in \{1, ..., k\}$, T is monotone on $\mathbb{R}^k$. It is not difficult to prove that $\mathrm{Fix}(T) = \emptyset$.

2) We consider Example 2.1.2 2). Take $y_0 \in \mathbb{R}^m$. On the set $\mathcal{F}(\mathbb{R}^k, \mathbb{R}^m)$, we define a mapping $T : \mathcal{F}(\mathbb{R}^k, \mathbb{R}^m) \to \mathcal{F}(\mathbb{R}^k, \mathbb{R}^m)$ by

$$T(f)(x) = 2f(x) + y_0,$$

for all $x \in \mathbb{R}^k$. It is easy to check that T is monotone on $(\mathcal{F}(\mathbb{R}^k, \mathbb{R}^m), \preceq_{\mathcal{F}})$. Clearly, $\mathrm{Fix}(T) = \{f : f(x) = -y_0,\ \forall x \in X\}$.

Definition 2.2.3 ([111]). A mapping $T : X \to X$ is said to be monotone sequentially continuous on X if for every monotone sequence $(x_n)_n$ of X that converges to some $x \in X$, the sequence $(T(x_n))_n$ converges to $T(x)$.

Obviously, continuous maps are monotone sequentially continuous. The converse is not true. Indeed, we can consider the following example.

Example 2.2.4. In $(\mathbb{R}^2, d)$ with the Euclidean metric d, we consider the partial order:

$$(u, v) \preceq (t, w) \Leftrightarrow (u = v \leq t = w) \text{ or } (u = t \text{ and } v = w).$$

The function g is defined as follows:

$$g(u, v) = \begin{cases} (1, 1) & \text{if } u \leq v \\ (0, 0) & \text{if } u > v \end{cases} \quad \text{for every } (u, v) \in \mathbb{R}^2.$$

Then g is monotone sequentially continuous on $\mathbb{R}^2$ but it is not continuous at any point $(u, u) \in \mathbb{R}^2$.

Example 2.2.5. Monotone nonexpansive mappings in metric spaces provide natural examples of monotone sequentially continuous mappings (see [105]).

We are going to present the first result concerning monotone mappings.

Theorem 2.2.6 ([103]). *Let Y be a nonempty closed bounded subset in a complete ordered metric space $(X, d, \preceq)$, and let ν be a measure of noncompactness on X. Let $T : Y \to Y$ be a monotone mapping satisfying*

$$\nu(T(\Omega)) \leq k\nu(\Omega)$$

for any $\Omega \subseteq Y$ with $k \in [0, 1)$. Assume that the set $\{y \in Y : y \preceq T(y)\}$ is nonempty. Then T has a fixed point.

Proof. Define $Y_0 = Y$, ..., $Y_{n+1} = \overline{T(Y_n)}$ for any $n \geq 0$. We have

$$\begin{aligned} \nu(Y_{n+1}) = \nu(\overline{T(Y_n)}) &= \nu(T(Y_n)) \leq k\nu(Y_n) \\ &\leq k^2\nu(Y_{n-1}) \leq \ldots \leq k^{n+1}\nu(Y_0). \end{aligned}$$

Consequently,

$$\lim_{n\to\infty} \nu(Y_{n+1}) = 0.$$

We have

$$\begin{aligned} Y_1 &= \overline{T(Y_0)} = \overline{T(Y)} \subseteq \overline{Y} = Y = Y_0, \\ Y_2 &= \overline{T(Y_1)} \subseteq \overline{T(Y_0)} = Y_1. \end{aligned}$$

By induction, we have that $Y_{n+1} \subseteq Y_n$ for any $n \geq 0$. By Definition 2.1.16 vi), we infer that the set $A = \bigcap\limits_{n=0}^{\infty} Y_n$ is a nonempty compact subset of Y.

For each $n \geq 0$, we have

$$T(Y_{n+1}) \subseteq T(Y_n) \subseteq \overline{T(Y_n)} = Y_{n+1}.$$

Therefore, it deduces that $T(A) \subseteq A$.

Take $x_0 \in \{y \in Y : y \preceq T(y)\}$. Put

$$A_n = \{T^n(x_0), T^{n+1}(x_0), \ldots\} \quad \text{for any } n \geq 0,$$

where $T^0(x_0) = x_0$. Note that for any $n \geq 0$, $A_n \subseteq Y_n$. Clearly, $T(A_n) = A_{n+1}$ for any $n \geq 0$. By an argument analogous to the one above, we get

$$\bigcap_{n=0}^{\infty} \overline{A_n} \neq \emptyset.$$

Take $c \in \bigcap\limits_{n=0}^{\infty} \overline{A_n} \subset A$. Then $c \in \overline{A_n}$ for any $n \geq 0$. If any subsequence of $(T^n(x_0))_n$ does not converge to c, then $c \in A_n$ for any $n \geq 0$. It implies that $c \preceq T(c)$.

Assume now that there exists a subsequence $(T^{n_k}(x_0))_k$ of $(T^n(x_0))_n$ such that $\lim\limits_{k\to\infty} T^{n_k}(x_0) = c$. Since $(T^{n_k}(x_0))_k$ is nondecreasing, it follows from Proposition 2.1.13 that

$$c = \sup_n T^n(x_0).$$

By monotonicity of T, we get

$$T^n(x_0) \preceq T(c) \quad \text{for each } n \geq 0,$$

and thus $c \preceq T(c)$.

Let $U = \{x \in A : x \preceq T(x)\}$. Since $c \in U$, $U \neq \emptyset$. Obviously, $T(x) \in U$ whenever $x \in U$. Suppose that Z is a chain in U. For each $z \in Z$, we set

$$V_z = [z, \rightarrow) \cap \overline{Z}.$$

Clearly, V_z is a nonempty closed subset of A for any $z \in Z$. Let $z_1, ..., z_n \in Z$. Since Z is a chain, there exists $i_0 \in \{1, ..., n\}$ with $z_{i_0} = \max\{z_1, ..., z_n\}$. It shows that $z_{i_0} \in V_{z_i}$ for all $i = 1, ..., n$. Thus

$$\bigcap_{i=1}^{n} V_{z_i} \neq \emptyset.$$

It means that the family $(V_z)_{z\in Z}$ has the finite intersection property. Hence

$$Z_0 = \bigcap_{z\in Z} V_z \neq \emptyset.$$

Take $v \in Z_0$. We have $z \preceq v$, so that $T(z) \preceq T(v)$ for all $z \in Z$. Hence $Z \subseteq (\leftarrow, T(v)]$. By the closedness of A and $(\leftarrow, T(v)]$, we deduce that $v \in \overline{Z} \subseteq (\leftarrow, T(v)] \cap A$. Thus $v \in U$ is an upper bound of Z in U. By Kuratowski-Zorn's lemma, there is a maximal element u^* in U. Furthermore, $u^* \preceq T(u^*) \in U$ from monotonicity of T. By maximality of u^* in U, we have $u^* = T(u^*)$. $\square$

Theorem 2.2.7 ([103]). *Let Y be a nonempty bounded closed subset of a complete ordered metric space $(X, d, \preceq)$, and let ν be a regular measure of noncompactness on X. Let $T : Y \to Y$ be a monotone mapping satisfying*

(i) $\nu(T(\Omega)) < \nu(\Omega)$ for any chain $\Omega \subset Y$ with $\nu(\Omega) > 0$, and

(ii) T is monotone sequentially continuous.

If $x_0 \in \{y \in Y : y \preceq T(y)\}$, then the iteration $(T^n(x_0))_n$ converges to a fixed point of T.

Proof. Assume that $x_0 \in Y$ such that $x_0 \preceq T(x_0)$. Let

$$\Omega = \{x_0, T(x_0), T^2(x_0), ...\}.$$

It is not difficult to prove that Ω is a chain. We have

$$\nu(\Omega) = \nu(T(\Omega) \cup \{x_0\}) = \nu(T(\Omega)).$$

It implies that $\nu(\Omega) = 0$, i.e., $\overline{\Omega}$ is compact. It follows from Proposition 2.1.15 that $\{T^n(x_0)\}$ converges to $y \in Y$. Since T is monotone sequentially continuous, $\lim\limits_{n\to\infty} T(T^n(x_0)) = T(y)$. We note that $(T(T^n(x_0)))_n$ is a subsequence of $(T^n(x_0))_n$. By the uniqueness of the limit point, we have $T(y) = y$. □

If we take $Y = [x_0, y_0]$, we obtain the following theorem. This theorem is a slight extension of Dhage's result [40] in the case of monotone single-valued mappings.

Theorem 2.2.8. *Let $(X, d, \preceq)$ be a complete ordered metric space, and let ν be a regular measure of noncompactness on X. Assume that x_0, y_0 belong to X such that $x_0 \preceq y_0$ and $x_0 \neq y_0$. Let $T : [x_0, y_0] \to [x_0, y_0]$ be a monotone mapping satisfying*

(i) $\nu(T(\Omega)) < \nu(\Omega)$ for any chain $\Omega \subset Y$ with $\nu(\Omega) > 0$, and

(ii) T is monotone sequentially continuous.

Then T has a minimal fixed point z_* and a maximal fixed point z^* in $[x_0, y_0]$, and

$$z^* = \lim_{n\to\infty} y_n, \qquad z_* = \lim_{n\to\infty} x_n,$$

where $y_n = T(y_{n-1})$, and $x_n = T(x_{n-1})$ for each $n = 1, 2, \ldots$.

Proof. By an argument analogous to that used in the proof of Theorem 2.2.7, the monotone sequence $(x_n)_n$ with $x_n = T(x_{n-1})$ for any $n = 1, 2, \ldots$ converges to $z_* \in [x_0, y_0]$, and

$$z_* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} T(x_n) = T(z_*).$$

We are going to prove that z_* is a minimal fixed point of T in $[x_0, y_0]$. Indeed, take $z \in [x_0, y_0]$ such that $T(z) = z$. Since T is monotone, it follows that $T(x_0) \preceq T(z)$, i.e., $x_1 \preceq z$. It implies that $x_2 = T(x_1) \preceq T(z) = z$. By induction, we have $x_n \preceq z$ for all $n \geq 1$. Since $(\leftarrow, z]$ is closed, it deduces that $z_* \preceq z$. Hence z_* is a minimal fixed point of T. Similarly, we can show the existence and maximality of the fixed point z^*. □

2.2.2 Common fixed points for a finite commutative family of monotone mappings

Definition 2.2.9. Let $\mathcal{F}$ be a family of mappings from X into X. We say that $\mathcal{F}$ is commutative if for any $S, T \in \mathcal{F}$, we have

$$T(S(x)) = S(T(x))$$

for any $x \in X$.

Let $\mathcal{F}$ be an arbitrary commutative family of monotone mappings. The recent result of Espínola-Wiśnicki [47] showed that in an ordered Hausdorff topological space X, if order intervals are compact then $\mathcal{F}$ has a common fixed point provided that there exists $x_0 \in X$ such that $x_0 \preceq F(x_0)$ for all $F \in \mathcal{F}$. Combining the approaches of Espínola–Wiśnicki and Sadovskiĭ [106], we show the existence of common fixed points for a commutative family $\mathcal{F}$.

Theorem 2.2.10. *Let Y be a nonempty bounded closed subset in a complete ordered metric space $(X, d, \preceq)$, and ν a regular measure of noncompactness on X. Let $\{T_1, \ldots, T_n\}$ be a nonempty finite commutative family of monotone maps from Y into Y satisfying*

$$\max\{\nu(T_i(\Omega)) : i = 1, \ldots, n\} < \nu(\Omega)$$

for any $\Omega \subseteq Y$ with $\nu(\Omega) > 0$. Assume that there exists $x_0 \in Y$ such that $x_0 \preceq T_i(x_0)$ for any $i \in \{1, \ldots, n\}$. Then $\bigcap_{i=1}^{n} \mathrm{Fix}(T_i) \neq \emptyset$.

Proof. Put

$$\mathcal{M} = \Big\{ M \subseteq Y : M \in \mathbb{CL}(X), x_0 \in M, \text{ and } T_i(M) \subseteq M \text{ for any } i \in \{1, \ldots, n\} \Big\}.$$

Clearly, $\mathcal{M} \neq \emptyset$ since $Y \in \mathcal{M}$. Set

$$A = \bigcap_{M \in \mathcal{M}} M, \quad \text{and} \quad B = \overline{\bigcup_{i=1}^{n} T_i(A) \cup \{x_0\}}.$$

Since $x_0 \in A$, A is a nonempty bounded closed set. It is not difficult to show that A belongs to $\mathcal{M}$, and so we get $T_i : A \to A$ for any $i \in \{1, \ldots, n\}$. Since $x_0 \in A$ and $\overline{T_i(A)} \subseteq \overline{A} = A$ for any $i \in \{1, \ldots, n\}$, it deduces that $B \subseteq A$. Thus

$$T_i(B) \subseteq T_i(A) \subseteq B \text{ for any } i \in \{1, \ldots, n\}.$$

It implies that $B \in \mathcal{M}$, hence $A \subseteq B$. Therefore, $B = A$. By properties of ν, we get

$$\begin{aligned} \nu(A) = \nu(B) &= \nu\big(\overline{\bigcup_{i=1}^{n} T_i(A) \cup \{x_0\}}\big) \\ &= \nu\big(\overline{\bigcup_{i=1}^{n} T_i(A)}\big) = \max\{\nu(T_i(A)) : i = 1, \ldots, n\}. \end{aligned}$$

It follows that $\nu(A) = 0$. Hence A is compact.

Let $U = \{x \in A : x \preceq T_i(x)$ for any $i \in \{1, \dots, n\}\}$. Since $x_0 \in U$, $U \neq \emptyset$. Since $\{T_1, \dots, T_n\}$ is a commutative family, $T_i(U) \subseteq U$ for any $i \in \{1, \dots, n\}$. Suppose that Z is a chain in U. For each $z \in Z$, we set

$$V_z = [z, \rightarrow) \cap \overline{Z}.$$

Clearly, V_z is a nonempty closed subset of A for any $z \in Z$. Let $z_1, ..., z_n \in Z$. Since Z is a chain, there exists $i_0 \in \{1, ..., n\}$ with $z_{i_0} = \max\{z_1, ..., z_n\}$. It shows that $z_{i_0} \in V_{z_i}$ for all $i = 1, ..., n$. Thus

$$\bigcap_{i=1}^{n} V_{z_i} \neq \emptyset.$$

It means that the family $(V_z)_{z \in Z}$ has the finite intersection property. Hence

$$Z_0 = \bigcap_{z \in Z} V_z \neq \emptyset.$$

Take $v \in Z_0$. For every $z \in Z$, we have $z \preceq v$, so that for any $i \in \{1, \dots, n\}$, $T_i(z) \preceq T_i(v)$. Hence $Z \subseteq (\leftarrow, T_i(v)]$. By the closedness of A and $(\leftarrow, T_i(v)]$, we deduce that $v \in \overline{Z} \subseteq (\leftarrow, T_i(v)] \cap A$. Thus $v \in U$ is an upper bound of Z in U. By Kuratowski-Zorn's lemma, there is a maximal element u^* in U. It deduces that for any $i \in \{1, \dots, n\}$, $x \preceq T_i(x) \preceq T_i(u^*)$ for every $x \in U$, $x \preceq u^*$. Note that $T_i(u^*) \in U$ for any $i \in \{1, \dots, n\}$. By maximality of u^*, we have $u^* = T_i(u^*)$ for any $i \in \{1, \dots, n\}$. Therefore, u^* is a common fixed point of $\{T_1, \dots, T_n\}$. □

When $T_1 = T_2 = \dots = T_n = T$, we obtain the following theorem.

Lemma 2.2.11. *Let Y be a nonempty bounded closed subset in a complete ordered metric space $(X, d, \preceq)$, and ν a regular measure of noncompactness on X. Let $T : Y \to Y$ be a monotone map satisfying*

$$\nu(T(\Omega)) < \nu(\Omega)$$

for any $\Omega \subseteq Y$ with $\nu(\Omega) > 0$. Assume that there exists $x_0 \in Y$ such that $x_0 \preceq T(x_0)$. Then $\mathrm{Fix}(T) \neq \emptyset$.

2.3. Fixed points of monotone multivalued mappings

It is natural to generalize Lemma 2.2.11 for monotone multivalued mappings. Before giving our result, we establish a lemma that appears to be interesting in its own right and will be used later.

Lemma 2.3.1 ([100]). *Let $(x_n)_n$ and $(y_n)_n$ be two sequences in an ordered metric space $(X, d, \preceq)$ that satisfy the following conditions:*

(i) $x_n \preceq x_{n+1}$ and $x_n \preceq y_n$ for every n;

(ii) $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$.

Then $x \preceq y$.

Proof. Since $\lim\limits_{n\to\infty} x_n = x$ and $(x_n)_n$ is monotone, we infer that $x = \sup\{x_n : n \geq 1\}$. Fix $n \geq 1$. It is not difficult to see that

$$y_m \in [x_n, \rightarrow) \text{ for all } m \geq n.$$

Since order intervals are closed, it implies that $y \in [x_n, \rightarrow)$ for every $n \geq 1$. Thus we have $x_n \in (\leftarrow, y]$ for any $n \geq 1$. Therefore, $x \preceq y$. □

Definition 2.3.2. Let $(X, \preceq)$ be a poset. A multivalued mapping $T : X \to 2^X \setminus \{\emptyset\}$ is called monotone if and only if for any $x, y \in X$ with $x \preceq y$ and any $x_1 \in T(x)$, there exists $y_1 \in T(y)$ such that $x_1 \preceq y_1$.

If $x \in T(x)$ then the point x is called a fixed point of T. The set of all fixed points of T is denoted by $\mathrm{Fix}(T)$.

In [74], Khamsi and Misane call a multivalued mapping $T : X \to 2^X \setminus \{\emptyset\}$ monotone if for any $x, y \in X$ with $x \preceq y$ and any $y_1 \in T(y)$, there exists $x_1 \in T(x)$ such that $x_1 \preceq y_1$.

In [42], Dhage refers to a map satisfying the conditions in Definition 2.3.2 as right monotone increasing. According to the definition provided by Khamsi and Misane, such maps are called left monotone increasing. A mapping that is both left and right monotone increasing is known as isotone increasing (see Definition 2.3, [40]).

Example 2.3.3. On a poset $(\mathcal{C}(I, \mathbb{R}), \preceq_c)$ (see Example 2.1.11), we define the multivalued mappings T_1 and T_2 from $\mathcal{C}(I, \mathbb{R})$ to $2^{\mathcal{C}(I,\mathbb{R})} \setminus \{\emptyset\}$ as follows:

$$T_1(f) = [f - 1, \rightarrow) \text{ and } T_2(f) = [f + 1, \rightarrow),$$

for every $f \in \mathcal{C}(I, \mathbb{R})$. Obviously, T_1, T_2 are monotone and $\mathrm{Fix}(T_1) = \mathcal{C}(I, \mathbb{R})$, $\mathrm{Fix}(T_2) = \emptyset$.

Theorem 2.3.4 ([100]). *Let Y be a nonempty bounded closed subset in a complete ordered metric space $(X, d, \preceq)$, and let ν be a regular measure of noncompactness on X. Let $T : Y \to \mathbb{CL}(Y)$ be a monotone multivalued mapping such that for each $\Omega \subseteq Y$ with $\nu(\Omega) > 0$, we have*

$$\nu(T(\Omega)) < \nu(\Omega),$$

where $T(\Omega) = \bigcup\limits_{x\in\Omega} T(x)$. Assume that $\{x \in Y : [x, \rightarrow) \cap T(x) \neq \emptyset\} \neq \emptyset$. Then T has a fixed point.

Proof. We are going to prove that there is a compact subset $A \subseteq Y$ such that $T(A) \subseteq A$. Take any $x_0 \in \{x \in Y : [x, \rightarrow) \cap T(x) \neq \emptyset\}$. Put

$$\mathcal{M} = \{M : M \in \mathbb{CL}(Y), x_0 \in M, \text{ and } T(M) \subseteq M\}.$$

Since $Y \in \mathcal{M}$, $\mathcal{M} \neq \emptyset$. We also set

$$A := \bigcap_{M\in\mathcal{M}} M, \quad \text{and } B := \overline{T(A)} \cup \{x_0\}.$$

Obviously, $A \neq \emptyset$ since $x_0 \in A$. It is not difficult to show that A belongs to $\mathcal{M}$, and so we have $T : A \to \mathbb{CL}(A)$. Moreover, $A = B$. Indeed, since $x_0 \in A$, $T(A) \subseteq A$, and A is closed, it deduces that $B \subseteq A$. Thus we have

$$T(B) \subseteq T(A) \subseteq B,$$

and so $B \in \mathcal{M}$. Hence $A \subseteq B$. By the properties of ν, we have

$$\nu(A) = \nu(B) = \nu(\overline{T(A)} \cup \{x_0\}) = \nu(\overline{T(A)}) = \nu(T(A)).$$

It deduces that $\nu(A) = 0$. Therefore, A is compact.

Put

$$U := \{x \in A : T(x) \cap [x, \to) \neq \emptyset\}.$$

Since $x_0 \in U$, U is nonempty. Note that for a fixed $x \in U$, we have $y \in U$ for all $y \in T(x)$ satisfying $x \preceq y$. Suppose that Z is a chain in U, and set

$$F_z := [z, \to) \cap \overline{Z} \quad \text{for each } z \in Z.$$

Clearly, F_z is a nonempty closed subset of A, for all $z \in Z$. Take any $z_1, ..., z_n \in Z$. Since Z is a chain, there exists $i_0 \in \{1, ..., n\}$ with $z_{i_0} = \max\{z_1, ..., z_n\}$. It deduces that $z_{i_0} \in F_{z_i}$ for all $i \in \{1, ..., n\}$. Consequently,

$$\bigcap_{i=1}^{n} F_{z_i} \neq \emptyset.$$

This means that the family $(F_z)_{z \in Z}$ has the finite intersection property. It implies that

$$Z_0 = \bigcap_{z \in Z} F_z \neq \emptyset.$$

Take $v \in Z_0$. Since Z is a chain, we can find a monotone sequence $(z_n)_n$ in Z such that $\lim_n z_n = v$. If $(z_n)_n$ is nonincreasing, then $v = \inf_n z_n$. Since $z_n \preceq v$, it implies that $z_n = v$ for any n. Hence $v \in U$. Assume now that $(z_n)_n$ is nondecreasing. Since $(z_n)_n \subseteq U$, there exists a sequence $(y_n)_n$ in A such that

$$z_n \preceq y_n \in T(z_n) \text{ for each } n \geq 1.$$

Since $z \preceq v$ for any $z \in Z$, $z_n \preceq v$ for any $n \geq 1$. By monotony of T, there is a sequence $(v_n)_n$ in $T(v)$ such that

$$y_n \preceq v_n \in T(v) \text{ for each } n \geq 1.$$

Note that $T(v)$ is compact. Thus we have $\lim_k v_{n_k} = t \in T(v)$ for a subsequence $(v_{n_k})_k$ of $(v_n)_n$. Now we have

$$z_{n_k} \preceq v_{n_k} \text{ for every } k \geq 1.$$

It follows from Lemma 2.3.1 that $v \preceq t \in T(v)$. Hence $v \in U$.

It deduces that v is an upper bound of Z in U. By Kuratowski-Zorn's lemma, U contains a maximal element u^*. Thus $u^* \preceq u$ for some $u \in T(u^*)$. Since $u \in U$, it implies that $u = u^*$. Therefore, u^* is a fixed point of T. □

Example 2.3.5. Define a multivalued map $T : [1,2] \to \mathbb{CL}([1,2])$ by

$$T(x) = \begin{cases} \frac{3}{2} & \text{if } x \in [1, \frac{5}{4}) \\ [\frac{3}{2}, \frac{5}{3}] & \text{if } x \in [\frac{5}{4}, \frac{3}{2}] \\ [\frac{5}{3}, 2] & \text{if } x \in (\frac{3}{2}, 2]. \end{cases}$$

It is easy to see that T is monotone on the poset $([1,2], \leq)$. Consider the Kuratowski measure of noncompactness α on $[1,2]$ defined in Example 2.1.17. Note that $\alpha(T(\Omega)) = \alpha(\Omega) = 0$ for any $\Omega \subset [1,2]$. We can easily see that $x_0 \in T(x_0)$ for any $x_0 \in \{\frac{3}{2}\} \cup [\frac{5}{3}, 2]$.

Example 2.3.6. Denote $c_0 = \{x = (x_n)_n : x_n \in \mathbb{R}, \lim\limits_{n\to\infty} x_n = 0\}$. On the vector space c_0, we consider the norm $\|x\|_{c_0} = \max\limits_{n\geq 1} |x_n|$ and the partial order defined by

$$x \preceq_{c_0} y \Leftrightarrow x_n \leq y_n \text{ for all } n \geq 1$$

for any $x = (x_n)_n$, $y = (y_n)_n \in c_0$. It is not difficult to show that $(c_0, \|.\|_{c_0}, \preceq_{c_0})$ is an ordered Banach space. The function ν defined by

$$\nu(\Omega) = \lim_{n\to\infty} \Big(\sup_{x\in\Omega} \Big(\max_{k\geq n} |x_k| \Big) \Big) \quad \text{for all } \Omega \in \mathbb{B}(c_0),$$

is a regular MNCs on c_0 (see [22]). Let $(t_n)_n$ be a real sequence such that

$$\inf_n t_n \geq \frac{1}{2}, \quad \text{and } \lim_{n\to\infty} t_n < 1.$$

We define a multivalued map T by

$$T(x) = \Big\{ y = (y_1, y_2, \ldots) : \frac{1}{2} x_n \leq y_n \leq t_n x_n \text{ for any } n \geq 1 \Big\}$$

for $x = (x_n)_n \in c_0$. Clearly, $T : \overline{B}(0,1) \to \mathbb{CL}(\overline{B}(0,1))$ and it is monotone. For each subset Ω of $\overline{B}(0,1)$ such that $\nu(\Omega) > 0$, we have

$$\begin{aligned} \nu(T(\Omega)) = & \lim_{n\to\infty} \Big(\sup_{y\in T(\Omega)} \Big(\max_{k\geq n} |y_k| \Big) \Big) \leq \lim_{n\to\infty} \Big(\sup_{x\in\Omega} \Big(\max_{k\geq n} |t_k x_k| \Big) \Big) \\ & < \lim_{n\to\infty} \Big(\sup_{x\in\Omega} \Big(\max_{k\geq n} |x_k| \Big) \Big) = \nu(\Omega). \end{aligned}$$

It is easy to see that $0 \in [0, \to) \cap T(0)$, where $0 = (0, 0, \ldots)$. Therefore, $0 \in \mathrm{Fix}(T)$.

2.4. Common fixed points for a commutative pair of monotone mappings

In this section, we are going to study the existence of common fixed points of a commutative pair of monotone mappings.

Definition 2.4.1. Let $(X, \preceq)$ be a poset. A multivalued $T : X \to 2^X \setminus \{\emptyset\}$ is called strongly monotone on X if for any $x, y \in X$ with $x \preceq y$ then for all $x_1 \in T(x)$, $y_1 \in T(y)$, we have $x_1 \preceq y_1$.

Clearly, if T is a strongly monotone mapping, T is also a monotone mapping. The term "strongly monotone multivalued mapping" is also used to refer to multivalued mappings that are of monotone nondecreasing type (I) (see Definition 2.5 (i), [112]).

Example 2.4.2. Let $(\mathbb{N}, \leq)$ be a poset. We define the multivalued mappings $F_1, F_2 : \mathbb{N} \to 2^{\mathbb{N}}$ as follows:

$$F_1(n) = \{n+1, n+2\}, \quad \text{and} \quad F_2(n) = \{n, n+1, n+2\}$$

for all $n \in \mathbb{N}$. It is not difficult to show that F_1 is a strongly monotone mapping. Clearly, F_2 is monotone but not strongly monotone.

Definition 2.4.3 ([107]). Let $(X, \preceq)$ be a poset. Let f be a mapping from X to X and F be a multivalued mapping from X to 2^X. The pair (f, F) is called commutative if $f(F(x)) = F(f(x))$ for all $x \in X$.

A point $x \in X$ is a common fixed point of the commutative pair (f, F) if $x = f(x) \in F(x)$.

Example 2.4.4. Define a monotone mapping $f : \mathbb{N} \to \mathbb{N}$ by $f(n) = n+1$ for each $n \in \mathbb{N}$, and consider the strongly monotone multivalued mapping F_1 in Example 2.4.2. Then we have

$$\begin{aligned} F_1(f(n)) &= F_1(n+1) = \{n+2, n+3\}, \\ f(F_1(n)) &= f(\{n+1, n+2\}) = \{n+2, n+3\} \end{aligned}$$

for each $n \in \mathbb{N}$. Hence the pair (f, F_1) is commutative.

Theorem 2.4.5. *Let Y be a nonempty bounded closed subset in a complete ordered metric space $(X, d, \preceq)$, and let ν be a regular measure of noncompactness on X. Let $F : Y \to \mathbb{CL}(Y)$ be a strongly monotone multivalued mapping, and $f : Y \to Y$ a monotone mapping such that $F(Y) \subseteq f(Y)$. Assume that the following conditions are satisfied:*

(i) (f, F) *is a commutative pair,*

(ii) for each $\Omega \subseteq Y$ with $\nu(\Omega) > 0$, we have

$$\max\{\nu(f(\Omega)), \nu(F(\Omega))\} < \nu(\Omega),$$

where $F(\Omega) = \bigcup_{x \in \Omega} F(x)$,

(iii) $\{x \in Y : x \preceq f(x)\} \cap \{x \in Y : x \preceq y \text{ for any } y \in F(x)\} \neq \emptyset$.

Then $\mathrm{Fix}(F) \cap \mathrm{Fix}(f) \neq \emptyset$.

Proof. By (iii), there exists $x_0 \in Y$ such that $x_0 \preceq f(x_0)$ and $x_0 \preceq y_0$ for any $y_0 \in F(x_0)$. Put

$$\mathcal{M} = \big\{M \subseteq Y : M \in \mathbb{CL}(X), x_0 \in M, \text{ and } f(M) \subseteq M, F(M) \subseteq M\big\}.$$

Since $Y \in \mathcal{M}$, $\mathcal{M}$ is nonempty. We also put

$$A = \bigcap_{M \in \mathcal{M}} M, \quad \text{and} \quad B = \overline{F(A) \cup f(A)} \cup \{x_0\}.$$

The set A is nonempty since $x_0 \in A$. It is not difficult to show that $A \in \mathcal{M}$. Since $x_0 \in A$, $F(A) \subseteq A$ and $f(A) \subseteq A$, it implies that $B \subseteq A$. Thus

$$F(B) \subseteq F(A) \subseteq B, \text{ and } f(B) \subseteq f(A) \subseteq B.$$

It deduces that $B \in \mathcal{M}$, so that $A \subseteq B$. Therefore, $B = A$. By the properties of measure of noncompactness, we have

$$\nu(A) = \nu(B) = \nu\big(\{x_0\} \cup \overline{F(A) \cup f(A)}\big) = \max\{\nu(f(A), \nu(F(A))\}.$$

It implies that $\nu(A) = 0$, hence A is compact. Put

$$\begin{aligned} V_f &:= \{x \in A : x \preceq f(x)\}, \\ V_F &:= \{x \in A : x \preceq y, \ \forall y \in F(x)\}. \end{aligned}$$

Clearly, V_f, V_F are nonempty, $f(V_f) \subseteq V_f$ and $F(V_F) \subseteq V_F$. We also put

$$V := V_f \cap V_F.$$

Since $x_0 \in V$, V is nonempty set. Now, we are going to show that $F(V_f) \subseteq V_f$, and $f(V_F) \subseteq V_F$.

For the first part, take $x \in V_f$ and $y \in F(x)$. Then we have $x \preceq f(x)$, and $f(y) \in f(F(x)) = F(f(x))$. By monotony of F, it deduces that $y \preceq f(y)$, i.e., $y \in V_f$. It shows that $F(x) \subseteq V_f$ for any $x \in V_f$. Therefore, $F(V_f) \subseteq V_f$.

Next, suppose that $x \in V_F$. For any $z \in F(f(x))$, there exists $t \in F(x)$ such that $z = f(t)$. Since $x \in V_F$, it implies that $x \preceq t$. Consequently, $f(x) \preceq f(t) = z$. Hence, $f(x) \preceq z$ for any $z \in F(f(x))$. Therefore, we have $f(V_F) \subseteq V_F$.

Finally, we get $f(V) \subseteq V$, $F(V) \subseteq V$. Suppose that Z is a chain in V. For each $z \in Z$, set

$$V_z = [z, \to) \cap \overline{Z}.$$

Clearly, V_z is a nonempty closed subset of C for any $z \in Z$. Let $z_1, ..., z_n \in Z$. Since Z is a chain, there exists $i_0 \in \{1, ..., n\}$ with $z_{i_0} = \max\{z_1, ..., z_n\}$. It shows that $z_{i_0} \in V_{z_i}$ for all $i = 1, ..., n$. Thus

$$\bigcap_{i=1}^{n} V_{z_i} \neq \emptyset.$$

It means that the family $(V_z)_{z \in Z}$ has the finite intersection property. Hence

$$Z_0 = \bigcap_{z \in Z} V_z \neq \emptyset.$$

Take $v \in Z_0$, so that $z \preceq v$ for any $z \in Z$. We have $f(z) \preceq f(v)$. It deduces that $z \preceq f(v)$ for any $z \in Z$, i.e., $Z \subseteq (\leftarrow, f(v)]$. By the closedness of ordered intervals, we have $\overline{Z} \subseteq (\leftarrow, f(v)] \cap A$. Therefore, $v \preceq f(v)$. On the other hand, by monotony of F, we get that for any $z \in Z$, $z_1 \in F(z)$ and $v_1 \in F(v)$, we have

$z_1 \preceq v_1$. Since $z \in V$, $z \preceq z_1$. It implies that $z \preceq v_1$ for any $z \in Z$. Hence $v \in (\leftarrow, v_1]$ for any $v_1 \in F(v)$, and so $v \in V$. Therefore, v is an upper bound of Z in V. By Kuratowski-Zorn's lemma, there is a maximal element u^* in V. It deduces that $x \preceq f(x) \preceq f(u^*)$ for every $x \in V$, $x \preceq u^*$. Hence $f(u^*)$ is an upper bound of $\{x \in V : x \preceq u^*\}$. Moreover, $f(u^*) \in V$. By maximality of u^*, we have $u^* = f(u^*)$. Furthermore, we get $u^* \preceq u$ for any $u \in F(u^*) \subseteq V$. It deduces that $u = u^*$. Therefore, u^* is a common fixed point of f and F. □

2.5. Some applications

In this section, we present some applications related to the existence of solutions of differential and integral equations.

2.5.1 An integral equation of Hammerstein type

Let us study the existence of solutions in $\mathcal{C}(I, \mathbb{R})$ of integral equations of the following form

$$f(x) = F(x, f(x)) + \int_0^1 H(x, s)U(s, f(s))ds \quad \text{for every } x \in I, \tag{2.5.1}$$

where $F : I \times \mathbb{R} \to \mathbb{R}$, $H : I \times I \to [0, \infty)$ are continuous, and $U(\cdot, f(\cdot))$ is Lebesgue measurable on I for each $f \in \mathcal{C}(I, \mathbb{R})$. By a solution of 2.5.1, we mean a function $f \in \mathcal{C}(I, \mathbb{R})$ such that

$$f(x) = F(x, f(x)) + \int_0^1 H(x, s)U(s, f(s))ds \quad \text{for every } x \in I.$$

Recall that in Example 2.1.11, we showed that $(\mathcal{C}(I, \mathbb{R}), \|\cdot\|_{\mathcal{C}}, \preceq_{\mathcal{C}})$ is an ordered Banach space, and by Example 2.1.2, we have $(I \times \mathbb{R}, \preceq_2)$ is a poset. In the proof of the following theorem, we use the regular MNCs Ψ_0 in Example 2.1.18.

Theorem 2.5.1 ([103]). *Assume that the functions in (2.5.1) satisfy the following:*

(i) $F(\cdot, \cdot)$ *is continuous on* $I \times \mathbb{R}$*, and* $F(x, \cdot)$ *is nondecreasing on* $\mathbb{R}$ *for every* $x \in I$*;*

(ii) there exists $k \in [0, 1)$ *such that*

$$|F(x, y) - F(x, z)| \le k|y - z| \quad \text{for any } x \in I \text{ and } y, z \in \mathbb{R};$$

(iii) $H(\cdot, \cdot)$ *is continuous on* $I \times I$*;*

(iv) $U(s, \cdot)$ *is nondecreasing on* $\mathbb{R}$ *for every* $s \in I$*, and* $U(\cdot, f(\cdot))$ *is Lebesgue measurable on* I *for each* $f \in \mathcal{C}(I, \mathbb{R})$*;*

(v) there exists a function $g : I \to [0, \infty)$ *such that*

$$\sup_{y \in \mathbb{R}} |U(s, y)| \le g(s) \quad \text{for a.e. } s \in I,$$

and

$$\int_0^1 g(s)ds < \infty;$$

(vi) there exists a function $f_0 \in \mathcal{C}(I,\mathbb{R})$ *such that*

$$f_0(x) \le F(x, f_0(x)) + \int_0^1 H(x,s)U(s, f_0(s))ds \quad \text{for every } x \in I. \quad (2.5.2)$$

Then the equation (2.5.1) has a solution $f = f(x)$ *belonging to* $\mathcal{C}(I;\mathbb{R})$.

Proof. We can assume that $\gamma := \int_0^1 g(s)ds > 0$. Take $f \in \mathcal{C}(I,\mathbb{R})$ and put

$$v(x) := \int_0^1 H(x,s)U(s, f(s))ds \quad \text{for all } x \in I.$$

Fix $s \in I$. Since $H(\cdot, s)$ is continuous on compact set I, for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $t, t' \in I$, $|t - t'| \le \delta$ we have

$$|H(t,s) - H(t',s)| \le \frac{\varepsilon}{\gamma}.$$

Hence

$$\begin{aligned} |v(t) - v(t')| &= \Big| \int_0^1 \Big(H(t,s) - H(t',s)\Big) U(s, f(s))ds \Big| \\ &\le \int_0^1 \Big|H(t,s) - H(t',s)\Big| |U(s, f(s))| ds \le \frac{\varepsilon}{\gamma} \int_0^1 g(s)ds = \varepsilon. \end{aligned}$$

Therefore, v is uniformly continuous on I. Now, fix a function $f \in \mathcal{C}(I,\mathbb{R})$ and put

$$T(f)(x) = F(x, f(x)) + \int_0^1 H(x,s)U(s, f(s))ds \quad \text{for all } x \in I.$$

Obviously, $T(f)$ is continuous on I. It implies that $T(\mathcal{C}(I,\mathbb{R})) \subseteq \mathcal{C}(I,\mathbb{R})$. Furthermore,

$$\begin{aligned} |T(f)(x)| &\le |F(x, f(x)) - F(x,0)| + |F(x,0)| + \int_0^1 |H(x,s)||U(s, f(s))|ds \\ &\le k|f(x)| + \|F(\cdot,0)\|_c + \gamma_1 \int_0^1 g(s)ds \end{aligned}$$

for any $x \in I$, where $\gamma_1 = \max\{H(x,s) : (x,s) \in I \times I\}$. Hence

$$\|T(f)\|_c \le k\|f\|_c + M,$$

where $M = \|F(\cdot,0)\|_c + \gamma_1\gamma$. It deduces that $T(\overline{B}_r(0)) \subseteq \overline{B}_r(0)$, where $r = M/(1-k)$.

Now take a nonempty subset Ω of $\overline{B}_r(0)$, $f \in \Omega$, and $\varepsilon > 0$. Choosing $x, y \in I$ such that $|x - y| \le \varepsilon$, we get

$$\begin{aligned} |T(f)(x) - T(f)(y)| &\le |F(x, f(x)) - F(y, f(y))| + \int_0^1 |H(x,s) - H(y,s)||U(s, f(s))|ds \\ &\le |F(x, f(x)) - F(x, f(y))| + |F(x, f(y)) - F(y, f(y))| \\ &\qquad + \int_0^1 |H(x,s) - H(y,s)||U(s, f(s))|ds \\ &\le k|f(x) - f(y)| + |F(x, f(y)) - F(y, f(y))| \\ &\qquad + \int_0^1 |H(x,s) - H(y,s)||U(s, f(s))|ds \\ &\le k\Psi(f,\varepsilon) + \overline{\Psi}_1(F,\varepsilon) + \gamma\overline{\Psi}_2(H,\varepsilon), \end{aligned}$$

where

$$\begin{aligned}\Psi(f,\varepsilon) &= \sup\{|f(x)-f(y)| : x,y\in I, |x-y|\le\varepsilon\},\\ \overline{\Psi}_1(F,\varepsilon) &= \sup\{|F(z,x)-F(w,x)| : z,w\in I, |z-w|\le\varepsilon, |x|\le r\},\\ \overline{\Psi}_2(H,\varepsilon) &= \sup\big\{\sup\{|H(x,s)-H(y,s)| : s\in I\} : x,y\in I, |x-y|\le\varepsilon\big\}.\end{aligned}$$

Since F, H are continuous respectively on compact sets $I\times[-r,r]$ and $I\times I$, then $\overline{\Psi}_1(F,\varepsilon)\to 0$ and $\overline{\Psi}_2(H,\varepsilon)\to 0$ as $\varepsilon\to 0$.

Thus

$$\Psi(T(f),\varepsilon)\le k\Psi(f,\varepsilon)+\overline{\Psi}_1(F,\varepsilon)+\gamma\overline{\Psi}_2(H,\varepsilon).$$

Hence

$$\Psi_0(T(\Omega))\le k\Psi_0(\Omega).$$

Clearly, T is monotone on $\overline{B}_r(0)$. Using Theorem 2.2.6, the equation 2.5.1 have a solution f in $\mathcal{C}(I,\mathbb{R})$. □

The following is an example of functional-integral equations satisfying our assumptions.

Example 2.5.2. Assume that $H : I\times I\to[0,\infty)$ is a continuous function. Then the following equation has a solution f in $\mathcal{C}(I,\mathbb{R})$:

$$f(x)=\frac{x(f(x)+1)}{1+x^2}+\int_0^1 H(x,s)\frac{f(s)}{\sqrt{s}(1+s)(1+|f(s)|)}ds.$$

2.5.2 An integral equation of Volterra type

In the following, we show the existence of solutions in $\mathcal{BC}(\mathbb{R}_+)$ of the functional-integral equation of the form

$$f(x)=F(f(x))+\int_0^x U(s,f(s))ds,\quad \text{for every } x\ge 0, \tag{2.5.3}$$

where $F:\mathbb{R}\to\mathbb{R}$ is continuous on $\mathbb{R}$, and for each $f\in\mathcal{BC}(\mathbb{R}_+)$, $x\ge 0$ the function $U(\cdot,f(\cdot))$ is Lebesgue measurable on $[0,x]$. By a solution of 2.5.3, we mean a function $f\in\mathcal{BC}(\mathbb{R}_+)$ such that

$$f(x)=F(f(x))+\int_0^x U(s,f(s))ds,\quad \text{for every } x\ge 0.$$

Firstly, we consider a relation on $\mathcal{BC}(\mathbb{R}_+)$ as follows:

$$f\preceq_{\mathcal{BC}} g \Leftrightarrow f(x)\le g(x) \text{ for all } x\in\mathbb{R}_+,$$

for every $f,g\in\mathcal{BC}(\mathbb{R}_+)$. It is easy to show that this realation is a partial order, so that $(\mathcal{BC}(\mathbb{R}_+),\preceq_{\mathcal{BC}})$ is a poset. We can prove that all order intervals are closed in the Banach space $(\mathcal{BC}(\mathbb{R}_+),\|\cdot\|_{\mathcal{BC}})$. Thus $(\mathcal{BC}(\mathbb{R}_+),\|\cdot\|_{\mathcal{BC}},\preceq_{\mathcal{BC}})$ is an ordered Banach space. Recall that $(\mathbb{R}\times\mathbb{R},\preceq_2)$ is a poset. In the proof of the following theorem, we use the MNCs Ψ_1 from Example 2.1.19.

Theorem 2.5.3 ([103]). *Assume that the following conditions are satisfied:*

(i) $F(\cdot)$ *is continuous and nondecreasing on* $\mathbb{R}$*;*

(ii) *there exists* $k \in [0,1)$ *such that*

$$|F(x) - F(y)| \le k|x - y| \quad \textit{for all } x, y \in \mathbb{R};$$

(iii) $U(s,\cdot)$ *is nondecreasing on* $\mathbb{R}$ *for every* $s \in \mathbb{R}_+$*, and for each* $f \in \mathcal{BC}(\mathbb{R}_+)$*,* $x \ge 0$*, the function* $U(\cdot, f(\cdot))$ *is Lebesgue measurable on* $[0, x]$*;*

(iv) *there exists a function* $g : [0,\infty) \to [0,\infty)$ *such that*

$$\sup_{x\in\mathbb{R}} |U(s,x)| \le g(s) \quad \textit{for a.e. } [0,+\infty),$$

and

$$\int_0^\infty g(s)ds < \infty;$$

(v) *there exists* $f_0 \in \mathcal{BC}(\mathbb{R}_+)$ *such that*

$$f_0(x) \le F(f_0(x)) + \int_0^x U(s, f_0(s))ds \quad \textit{for every } x \ge 0. \tag{2.5.4}$$

Then the equation (2.5.3) has a solution f *belonging to the space* $\mathcal{BC}(\mathbb{R}_+)$*.*

Proof. Fix a function $f \in \mathcal{BC}(\mathbb{R}_+)$ and put

$$T(f)(x) = F(f(x)) + \int_0^x U(s, f(s))ds \quad \text{for all } x \ge 0.$$

Note that the function $u(x) := \int_0^x U(s, f(s))ds$ is continuous on $[0,\infty)$, and the function $F(f)$ is continuous on $[0,\infty)$. Furthermore,

$$\begin{aligned} |T(f)(x)| &\le |F(f(x)) - F(0)| + |F(0)| + \int_0^x |U(s, f(s))|ds \\ &\le k|f(x) - 0| + |F(0)| + \int_0^x g(s)ds \\ &\le k|f(x)| + |F(0)| + \int_0^\infty g(s)ds \end{aligned}$$

for every $x \in \mathbb{R}_+$. Hence $T(f) \in \mathcal{BC}(\mathbb{R}_+)$. It means that $T(\mathcal{BC}(\mathbb{R}_+)) \subseteq \mathcal{BC}(\mathbb{R}_+)$.

Now, we have

$$\|T(f)\|_{\mathcal{BC}} \le k\|f\|_{\mathcal{BC}} + |F(0)| + \int_0^\infty g(s)ds.$$

Therefore, if $\|f\|_{\mathcal{BC}} \le r$ with $r = \frac{R}{1-k}$, where $R = |F(0)| + \int_0^\infty g(s)ds$, then $\|T(f)\|_{\mathcal{BC}} \le r$. It shows that $T(\overline{B}_r(0)) \subseteq \overline{B}_r(0)$. Moreover, T is monotone on $\overline{B}_r(0)$.

Take a nonempty subset Ω of $\overline{B}_r(0)$, a function $f \in \Omega$. Fix $L > 0$, $\varepsilon > 0$. Choosing $z, w \in [0, L]$ such that $|z - w| \leq \varepsilon$, we get

$$\begin{aligned}
|T(f)(z) - T(f)(w)| &\leq |F(f(z)) - F(f(w))| + \Big| \int_z^w U(s, f(s)ds \Big| \\
&\leq k|f(z) - f(w)| + \Big| \int_z^w |U(s, f(s))|ds \Big| \\
&\leq k|f(z) - f(w)| + \Big| \int_z^w g(s)ds \Big| \\
&\leq k\Psi^L(f, \varepsilon) + \sup \Big\{ \Big| \int_z^w g(s)ds \Big| : z, w \in [0, L], |z - w| \leq \varepsilon \Big\}.
\end{aligned}$$

Thus

$$\Psi^L(T(f), \varepsilon) \leq k\Psi^L(f, \varepsilon) + \sup \Big\{ \Big| \int_z^w g(s)ds \Big| : z, w \in [0, L], |z - w| \leq \varepsilon \Big\}.$$

It yields

$$\Psi^L(T(\Omega), \varepsilon) \leq k\Psi^L(\Omega, \varepsilon) + \sup \Big\{ \Big| \int_z^w g(s)ds \Big| : z, w \in [0, L], |z - w| \leq \varepsilon \Big\}.$$

Using our assumptions, it is not difficult to prove that the function

$$h(t) = \int_0^t g(s)ds$$

is uniformly continuous on the compact set $[0, L]$. Hence

$$\lim_{\varepsilon \to 0} \sup \Big\{ \Big| \int_z^w g(s)ds \Big| : z, w \in [0, L], |z - w| \leq \varepsilon \Big\} = 0.$$

By this estimate, we get

$$\Psi_0^L(T(\Omega)) \leq k\Psi_0^L(\Omega). \tag{2.5.5}$$

Now, choose $z, w \in [L, \infty)$, we get

$$|T(f)(z) - T(f)(w)| \leq k|f(z) - f(w)| + \Big| \int_z^w g(s)ds \Big|.$$

Hence

$$\Phi(T(f), L) \leq k\Phi(f, L) + \overline{\Phi}(g, L),$$

where

$$\begin{aligned}
\Phi(T(f), L) &= \sup\{|T(f)(z) - T(f)(w)| : z, w \in [L, \infty)\}, \\
\Phi(f, L) &= \sup\{|f(z) - f(w)| : z, w \in [L, \infty)\}, \\
\overline{\Phi}(g, L) &= \sup \Big\{ \Big| \int_z^w g(s)ds \Big| : z, w \in [L, \infty) \Big\}.
\end{aligned}$$

It implies that

$$\sup_{f \in \Omega} \Phi(T(f), L) \leq k \sup_{f \in \Omega} \Phi(f, L) + \overline{\Phi}(g, L).$$

Now, we observe that the function defined by

$$\eta(U) = \int_U g(s)ds$$

for $U \in \mathcal{B}(\mathbb{R}_+)$ is a measure on $\mathcal{B}(\mathbb{R}_+)$, where $\mathcal{B}(\mathbb{R}_+)$ is the family of Borel subsets of $\mathbb{R}_+$. Take $z, w \in [L, \infty)$. Since $g(s) \geq 0$ for almost $s \geq 0$, we have

$$\Big| \int_z^w g(s)ds \Big| \leq \int_T^\infty g(s)ds = \eta([L, \infty)).$$

Hence

$$\sup \Big\{ \Big| \int_z^w g(s)ds \Big| : z, w \in [L, \infty) \Big\} \leq \eta([L, \infty)),$$

and thus

$$\begin{aligned}\lim_{L\to\infty} \sup \Big\{ \Big| \int_z^w g(s)ds \Big| : z, w \in [L, \infty) \Big\} \leq & \lim_{L\to\infty} \eta([L, \infty)) \\ \leq & \eta\Big(\bigcap_{\alpha \geq L} [\alpha, \infty) \Big) = \eta(\emptyset) = 0.\end{aligned}$$

Therefore,

$$b(T(\Omega)) \leq kb(\Omega). \tag{2.5.6}$$

It follows from (2.5.5) and (2.5.6) that

$$\Psi_1(T(\Omega)) \leq k\Psi_1(\Omega),$$

where

$$\Psi_1(\Omega) = \Psi_0(\Omega) + b(\Omega).$$

By Theorem 2.2.6, there exists a solution f of (2.5.3) in $\mathcal{BC}(\mathbb{R}_+)$. □

The following are examples satisfying the assumptions of Theorem 2.5.3:

Example 2.5.4.

$$f(x) = \arctan \frac{f(x)+2}{2} + \frac{1}{2} \int_0^x \frac{f(s)\sin^2 s}{s^2\sqrt{f^2(s)+1}} ds,$$

$$2f(x) = \frac{f(x)}{1+|f(x)|} + \frac{3}{\pi^2 \log 2} \int_0^x \frac{s}{\sqrt{e^s - 1}} \arctan f(s) ds.$$

2.5.3 A first order initial value problem with discontinuities

Let $(E, \|\cdot\|_E, \preceq_E)$ be an ordered Banach space with the partial order $\preceq_E$. Take a positive number r, and put $\overline{B}_r = \overline{B}_r(0) \subset E$. Let $\mathcal{C}(I, E)$ denote the space of all mappings $x : I \to E$ that are continuous on I. Define the supremum norm on $\mathcal{C}(I, E)$:

$$\|x\|_{\mathcal{C}_E} = \sup\{\|x(t)\|_E : t \in I\}.$$

We check at once that $(\mathcal{C}(I, E), \|\cdot\|_{\mathcal{C}_E})$ is a Banach space. Define a relation on $\mathcal{C}(I, E)$ by

$$x \preceq_{\mathcal{C}_E} y \Leftrightarrow x(t) \preceq_E y(t) \text{ for every } t \in I,$$

for every $x, y \in \mathcal{C}(I, E)$. It is not difficult to prove that $\preceq_{\mathcal{C}_E}$ is a partial order on $\mathcal{C}(I, E)$, and order intervals in $\mathcal{C}(I, E)$ are closed sets in the Banach space $(\mathcal{C}(I, E), \|\cdot\|_{\mathcal{C}_E})$. Therefore, $(\mathcal{C}(I, E), \|\cdot\|_{\mathcal{C}_E}, \preceq_{\mathcal{C}_E})$ is an ordered Banach space.

Next, consider the Kuratowski MNCs α on the space $\mathcal{C}(I, E)$. The following result is proved in [113, Theorem II.2.11].

Theorem 2.5.5. *Assume that Ω is an equicontinuous bounded subset of $\mathcal{C}(I, E)$. Then*

$$\alpha(\Omega) = \sup_{t \in I} \alpha(\{x(t) : x \in \Omega\}).$$

We are concerned with the solvability of the following first-order initial value problem

$$x'(t) = f(t, x(t)) \text{ for a.e. } t \in I, \text{ and } x(0) = 0, \tag{2.5.7}$$

where $f : I \times \overline{B}_r \to \overline{B}_r$.

Before going to our next theorem, let us recall the following lemma (see [90, Lemma II.1.3]).

Lemma 2.5.6. *If $h : I \to E$ is integrable then*

$$\int_0^t h(s)ds \in t\overline{\text{co}}(\{h(s) : s \in [0, t]\})$$

for all $t \in I$.

Theorem 2.5.7 ([103]). *Assume that $f : I \times \overline{B}_r \to \overline{B}_r$ satisfies the following conditions:*

(i) $f(t, \cdot)$ is nondecreasing on $\overline{B}_r$ for almost all $t \in I$, and

$$M = \sup\{\|f(t, x)\|_E : (t, x) \in I \times \overline{B}_r\};$$

(ii) $f(\cdot, x(\cdot))$ is integrable on I for every $x \in \mathcal{C}(I, \overline{B}_r)$;

(iii) there exists $k \geq 0$ such that

$$\alpha(f(I \times \Omega)) \leq k\alpha(\Omega) \text{ for any } \Omega \subseteq \overline{B}_r;$$

(iv) there exists a function $x_0 \in \overline{B}_r$ such that

$$x_0(t) \preceq \int_0^t f(s, x_0(s))ds \quad \text{for every } t \in I.$$

Then the equation (2.5.7) has a solution $x \in \mathcal{C}(J, \overline{B}_r)$, where $J = [0, \beta]$ with $0 < \beta < \min\{1, r/M, 1/k\}$.

Proof. Rather than (2.5.7) we consider the following integral equation

$$x(t) = \int_0^t f(s, x(s))ds. \tag{2.5.8}$$

Fix $x \in \mathcal{C}(J, \overline{B}_r)$, and put

$$T(x)(t) = \int_0^t f(s, x(s))ds.$$

Since $f(\cdot, x(\cdot))$ is integrable, then $T(x)$ is continuous on J. For $z, w \in J$, we get

$$\|T(x)(z) - T(x)(w)\|_E = \left\| \int_z^w f(s, x(s))ds \right\|_E \leq M\|z - w\|_E,$$

and

$$\|T(x)(z)\|_E \leq \beta M.$$

It means that $T(\mathcal{C}(J, \overline{B}_r)) \subseteq \mathcal{C}(J, \overline{B}_r)$, and $T(\mathcal{C}(J, \overline{B}_r))$ is equicontinuous and bounded. Take $\Omega \subseteq \mathcal{C}(J, \overline{B}_r)$ such that $\alpha(\Omega) > 0$. By Theorem 2.5.5, we get

$$\begin{aligned}\alpha(T(\Omega)) &= \sup\Big\{\alpha(\{(Tx)(t) : x \in \Omega\}) : t \in J\Big\} \\ &= \sup\Big\{\alpha\Big(\Big\{\int_0^t f(s, x(s))ds : x \in \Omega\Big\}\Big) : t \in J\Big\}.\end{aligned}$$

From Lemma 2.5.6, it deduces that

$$\begin{aligned}\alpha(T(\Omega)) &\leq \sup_{t\in J}\{\alpha(t\overline{\text{co}}(\{f(s, x(s)) : s \in [0, t], x \in \Omega\}))\} \\ &\leq \sup_{t\in J}\{t\alpha(\overline{\text{co}}(f(J \times \Omega)))\} \leq \beta\alpha(f(J \times \Omega)) \leq \beta k\alpha(\Omega) < \alpha(\Omega).\end{aligned}$$

Obviously, T is a monotone operator on $\mathcal{C}(J, \overline{B}_r)$. Note that $x_0 \preceq_{\mathcal{C}_E} Tx_0$. Lemma 2.2.11 yields that T has a fixed point $x \in \mathcal{C}(J, \overline{B}_r)$ that is a solution of equation (2.5.7). □

2.5.4 Functional Integral Inclusion

Denote all Lebesgue integrable functions defined on I by $L^1(I, \mathbb{R})$. This space is equipped with the following norm

$$\|g\|_1 = \int_0^1 g d\mu,$$

for every $g \in L^1(I, \mathbb{R})$. We can show that $(L^1(I, \mathbb{R}), \|\cdot\|_1)$ is a Banach space.

In this section, we prove the existence of solutions to a functional integral inclusion in the following form

$$f(x) \in F(x, f(x)) + \int_0^x k(x, s)\mathcal{F}(s, f(s))ds, \quad \text{for every } x \in I, \tag{2.5.9}$$

where $F : I \times \mathbb{R} \to \mathbb{R}$, $k : I \times I \to \mathbb{R}$ are continuous, and $\mathcal{F} : I \times \mathbb{R} \to \mathbb{CL}(\mathbb{R})$. By solution of 2.5.9, we mean a function $f \in \mathcal{C}(I, \mathbb{R})$ such that

$$f(x) = F(x, f(x)) + \int_0^x k(x, s)f_1(s)ds, \quad \text{for every } x \in I,$$

where $f_1(\cdot) \in \mathcal{F}(\cdot, f(\cdot))$ and $f_1 \in L^1(I, \mathbb{R})$. Firstly, we recall some basic definitions used in this section.

Definition 2.5.8. Let (X, d) be a metric space, and $T : X \to 2^X$ be a multivalued mapping on X.

(i) T is called upper semi-continuous if for any open subset Y of X, the set $\{x \in X : T(x) \subseteq Y\}$ is open in X.

(ii) T is said to be totally bounded if for any $B \in \mathbb{B}(X)$, $\overline{T(B)}$ is a totally bounded subset of X.

(iii) T has a closed graph if for $\lim_{n\to\infty} x_n = x^*$, $\lim_{n\to\infty} y_n = y^*$ and $y_n \in T(x_n)$, we have $y^* \in T(x^*)$.

Note that if a multivalued map T is totally bounded with nonempty compact values, then T is upper semi-continuous if and only if T has a closed graph.

Definition 2.5.9. A multivalued map $\mathcal{F} : I \times \mathbb{R} \to \mathbb{CP}(\mathbb{R})$ is said to be L^1-Carathéodory if

(i) for each $x \in \mathbb{R}$, the mapping $\mathcal{F}(\cdot, x)$ is measurable,

(ii) for almost all $t \in I$, the mapping $\mathcal{F}(t, \cdot)$ is upper semi-continuous,

(iii) for each $\rho > 0$, there exists a function $g_\rho \in L^1(I, \mathbb{R}_+)$ such that for all $u \in \mathbb{R}$ with $|u| \leq \rho$,

$$|||\mathcal{F}(t, u)||| = \sup\{|v| : v \in \mathcal{F}(t, u)\} \leq g_\rho(t), \ \textit{a.e. } t \in I.$$

For any function $f \in \mathcal{C}(I, \mathbb{R})$, consider the selection set

$$S_{\mathcal{F}}(f) = \{f_1 \in L^1(I, \mathbb{R}) : f_1(s) \in \mathcal{F}(s, f(s)), \ \text{a.e. } s \in I\}.$$

In [82], Lasota and Opial showed that if $\mathcal{F}$ is L^1-Carathéodory, then $S_{\mathcal{F}}(f) \neq \emptyset$ for each $f \in \mathcal{C}(I, \mathbb{R})$. They also established the following lemma.

Lemma 2.5.10. *Assume that a multivalued map $\mathcal{F}$ satisfies the conditions (i), (ii) of Definition 2.5.9 with $S_{\mathcal{F}}(f) \neq \emptyset$ for each $f \in \mathcal{C}(I, \mathbb{R})$. Let $\mathcal{G} : L^1(I, \mathbb{R}) \to \mathcal{C}(I, \mathbb{R})$ be a continuous linear mapping. Then $\mathcal{G} \circ S_{\mathcal{F}} : \mathcal{C}(I, \mathbb{R}) \to 2^{\mathcal{C}(I,\mathbb{R})}$ is a closed graph operator on $\mathcal{C}(I, \mathbb{R}) \times \mathcal{C}(I, \mathbb{R})$.*

Now we present our main theorem for this section.

Theorem 2.5.11 ([100]). *Assume that the maps in the functional integral inclusion 2.5.9 satisfy the following conditions:*

(C1) $F(\cdot, \cdot)$ is continuous on $I \times \mathbb{R}$, and $F(t, \cdot)$ is nondecreasing for every $t \in I$;

(C2) there exists $L \in [0, 1)$ such that

$$|F(x, f) - F(x, g)| \leq L|f - g|, \quad \textit{for each } f, g \in \mathbb{R}, \ x \in I;$$

(C3) $k(\cdot, \cdot)$ is continuous on $I \times I$;

(C4) $\mathcal{F} : I \times \mathbb{R} \to \mathbb{CP}(\mathbb{R})$ is L^1-Carathéodory;

(C5) $S_{\mathcal{F}}(\cdot)$ *is monotone: for any* $f, g \in \mathcal{C}(I,\mathbb{R})$ *with* $f \preceq_{\mathcal{C}} g$ *and any* $f_1 \in S_{\mathcal{F}}(f)$, *there is* $g_1 \in S_{\mathcal{F}}(g)$ *such that* $f_1(s) \leq g_1(s)$ *for a.e.* $s \in I$*;*

(C6) there exists a positive number r *such that*

$$r \geq \frac{\|F(x,0)\|_{\mathcal{C}} + M\|g_r\|_1}{1-L},$$

where $M = \max\{|k(x,y)| : (x,y) \in I \times I\}$, *the function* g_r *satisfies Definition 2.5.9 (iii);*

(C7) there exists $f_0 \in \mathcal{C}(I,\mathbb{R})$ *such that* $f_0 \preceq_{\mathcal{C}} h_0$ *for some* $h_0 \in \mathcal{C}(I,\mathbb{R})$ *with*

$$h_0(x) \in F(x, f_0(x)) + \int_0^x k(x,s)\mathcal{F}(s, f_0(s))ds, \quad \textit{for every } x \in I.$$

Then the integral inclusion 2.5.9 has at least one solution in $\mathcal{C}(I,\mathbb{R})$.

Proof. Take $f \in \mathcal{C}(I,\mathbb{R})$, and put

$$\mathcal{T}(f)(x) = F(x, f(x)) + \int_0^x k(x,s)\mathcal{F}(s, f(s))ds, \quad \text{for every } x \in I. \qquad (2)$$

We recall the following basic result: if $f_1 \in L^1(I,\mathbb{R})$, then the function

$$F_1(x) = \int_0^x k(x,s)f_1(s)ds$$

is continuous on I. It implies that the function

$$F_2(x) = F(x, f(x)) + F_1(x) = F(x, f(x)) + \int_0^x k(x,s)f_1(s)ds$$

is continuous on I for any $f_1 \in S_{\mathcal{F}}(f)$. Hence for each $f \in \mathcal{C}(I,\mathbb{R})$, we have $\mathcal{T}(f) \subseteq \mathcal{C}(I,\mathbb{R})$.

Next, we are going to show that $\mathcal{T}(f)$ is closed for each $f \in \mathcal{C}(I,\mathbb{R})$. Let $(h_n)_n$ be a sequence in $\mathcal{T}(f)$ and $h_0 \in \mathcal{C}(I,\mathbb{R})$ such that $\|h_n - h_0\|_{\mathcal{C}} \to 0$ as $n \to \infty$. We need to show that $h_0 \in \mathcal{T}(f)$. Since $h_n \in \mathcal{T}(f)$, there exists $f_n \in S_{\mathcal{F}}(f)$ such that

$$h_n(x) = F(x, f(x)) + \int_0^x k(x,s)f_n(s)ds, \quad \text{for every } x \in I.$$

Consider the operator $\mathcal{G} : L^1(I,\mathbb{R}) \to \mathcal{C}(I,\mathbb{R})$ defined by

$$\mathcal{G}(f)(x) = \int_0^x k(x,s)f(s)ds, \quad \text{for every } x \in I.$$

Obviously, $\mathcal{G}$ is continuous and linear. It follows from Lemma 2.5.10 that $\mathcal{G} \circ S_{\mathcal{F}}$ is a closed graph operator on $\mathcal{C}(I,\mathbb{R}) \times \mathcal{C}(I,\mathbb{R})$. Furthermore, since $\max_{x\in I} |(h_n(x) - F(x, f(x)) - (h_0(x) - F(x, f(x))| \to 0$ as $n \to \infty$, and $h_n(.) - F(., f(.)) \in \mathcal{G} \circ S_{\mathcal{F}}(f)$, we have

$$h_0(.) - F(., f(.)) \in \mathcal{G} \circ S_{\mathcal{F}}(f).$$

It implies that there is $f_0 \in S_{\mathcal{F}}(f)$ such that

$$h_0(x) - F(x, f(x)) = \int_0^x k(x,s) f_0(s) ds, \quad x \in I.$$

Therefore, $h_0 \in \mathcal{T}(f)$.

Next, we are going to prove that $\mathcal{T} : \overline{B}_r(0) \to \mathbb{CL}(\overline{B}_r(0))$. Take $f \in \overline{B}_r(0)$ and $h \in \mathcal{T}(f)$. Then there is $h_1 \in S_{\mathcal{F}}(f)$ such that

$$h(x) = F(x, f(x)) + \int_0^x k(x,s) h_1(s) ds, \quad \text{for every } x \in I.$$

We have

$$\begin{aligned} |h(x)| &\leq |F(x, f(x)) - F(x,0)| + |F(x,0)| + \left| \int_0^x k(x,s) h_1(s) ds \right| \\ &\leq L|f(x)| + \|F(x,0)\|_c + \int_0^x |k(x,s)| \|| \mathcal{F}(s, f(s)) \|| ds \\ &\leq L\|f\|_c + \|F(x,0)\|_c + M\|g_r\|_1 \leq r \end{aligned}$$

for every $x \in I$. It implies that $h \in \overline{B}_r(0)$. Hence $\mathcal{T}(f) \in \mathbb{CL}(\overline{B}_r(0))$ for every $f \in \overline{B}_r(0)$.

Take $f, h \in \overline{B}_r(0)$ such that $f \preceq_{\mathcal{C}} h$. By (C1),

$$F(x, f(x)) \leq F(x, h(x)) \text{ for all } x \in I.$$

Furthermore, for each $f_1 \in \mathcal{T}(f)$, there exists $f_2 \in S_{\mathcal{F}}(f)$ such that

$$f_1(x) = F(x, f(x)) + \int_0^x k(x,s) f_2(s) ds, \quad \text{for every } x \in I.$$

By (C5), there is $h_2 \in S_{\mathcal{F}}(h)$ such that $f_2(s) \leq h_2(s)$ for a.e. $s \in I$. Put

$$h_1(x) = F(x, h(x)) + \int_0^x k(x,s) h_2(s) ds, \quad \text{for every } x \in I.$$

Clearly, $h_1 \in \mathcal{T}(h)$ and $f_1(x) \leq h_1(x)$ for every $x \in I$. Hence $\mathcal{T}$ is monotone on $\overline{B}_r(0)$.

Now assume that Ω is a nonempty subset of $\overline{B}_r(0)$ and $f \in \Omega$. Take any function $f_1 \in \mathcal{T}(f)$. Then there exists $f_2 \in S_{\mathcal{F}}(f)$ such that

$$f_1(x) = F(x, f(x)) + \int_0^x k(x,s) f_2(s) ds, \quad \text{for every } x \in I.$$

Fix $\varepsilon > 0$ and choose $x, y \in I$ such that $|x - y| \leq \varepsilon$, we get

$$\begin{aligned}
|f_1(x) - f_1(y)| &\le |F(x, f(x)) - F(y, f(y))| + \Big| \int_0^x k(x,s) f_2(s) ds - \int_0^y k(y,s) f_2(s) ds \Big| \\
&\le |F(x, f(x)) - F(x, f(y))| + |F(x, f(y)) - F(y, f(y))| \\
&\quad + \Big| \int_0^x k(x,s) f_2(s) ds - \int_0^x k(y,s) f_2(s) ds \Big| \\
&\qquad + \Big| \int_0^x k(y,s) f_2(s) ds - \int_0^y k(y,s) f_2(s) ds \Big| \\
&\le L|f(x) - f(y)| + |F(x, f(y)) - F(y, f(y))| \\
&\qquad + \int_0^x |k(x,s) - k(y,s)||f_2(s)| ds + \Big| \int_x^y |k(y,s)||f_2(s)| ds \Big| \\
&\le L|f(x) - f(y)| + |F(x, f(y)) - F(y, f(y))| \\
&\qquad + \int_0^x |k(x,s) - k(y,s)| g_r(s) ds + M \Big| \int_x^y g_r(s) ds \Big| \\
&\le L|f(x) - f(y)| + |F(x, f(y)) - F(y, f(y))| \\
&\qquad + \int_0^1 |k(x,s) - k(y,s)| g_r(s) ds + M|q(x) - q(y)|,
\end{aligned}$$

where

$$q(x) = \int_0^x g_r(s) ds.$$

Using given assumptions, we infer that the function $F(z,t)$ is uniformly continuous on $I \times [-r, r]$, and the function $q(x)$ is uniformly continuous on I. Hence, when $\varepsilon \to 0$, we have

$$\begin{aligned}
\Psi_r(F, \varepsilon) &:= \sup\{|F(x,z) - F(y,z)| : x, y \in I, |x - y| \le \varepsilon, |z| \le r\} \to \ 0, \\
\Psi_r(k, g_r, \varepsilon) &:= \sup\Big\{ \int_0^1 |k(x,s) - k(y,s)| g_r(s) ds : x, y \in I, |x - y| \le \varepsilon \Big\} \to \ 0, \\
\overline{\Psi}(q, \varepsilon) &:= \sup\{|q(x) - q(y)| : x, y \in I, |x - y| \le \varepsilon\} \to \ 0.
\end{aligned}$$

Now, from the obtained estimate, we have

$$\Psi(f_1, \varepsilon) \le L\Psi(f, \varepsilon) + \Psi_r(F, \varepsilon) + \Psi_r(k, g_r, \varepsilon) + \overline{\Psi}(q, \varepsilon).$$

It yields

$$\begin{aligned}
\Psi(\mathcal{T}(\Omega), \varepsilon) = \sup_{f_1 \in T(\Omega)} \Psi(f_1, \varepsilon) &\le L \sup_{f \in \Omega} \Psi(f, \varepsilon) + \Psi_r(F, \varepsilon) + \Psi_r(k, g_r, \varepsilon) + \overline{\Psi}(q, \varepsilon) \\
&\le L\Psi(\Omega, \varepsilon) + \Psi_r(f, \varepsilon) + \Psi_r(k, g_r, \varepsilon) + \overline{\Psi}(q, \varepsilon),
\end{aligned}$$

and consequently,

$$\Psi_0(\mathcal{T}(\Omega)) \le L\Psi_0(\Omega) < \Psi_0(\Omega).$$

It follows that the mapping $\mathcal{T}$ satisfies all conditions of Theorem 2.3.4. Therefore, the functional integral inclusion (2.5.9) admits a solution in $\mathcal{C}(I, \mathbb{R})$. □

Chapter 3

Fixed points of G-monotone mappings in Banach spaces

3.1. Preliminaries

It is often preferable to investigate graphs rather than confining ourselves to partial orders. Jachymski [58] adopted this approach, extending the Banach contraction principle to metric spaces equipped with a graph.

In this section, we introduce some basic definitions in graph theory.

Definition 3.1.1 ([23, 45, 66]). A graph G is a pair $(V(G), E(G))$, where the elements of a nonempty set $V(G)$ are called vertices of G, and $E(G)$ is a set of paired vertices called edges. If a direction is imposed on each edge, we call it a directed graph or digraph.

Example 3.1.2. We consider the following graph and digraph.

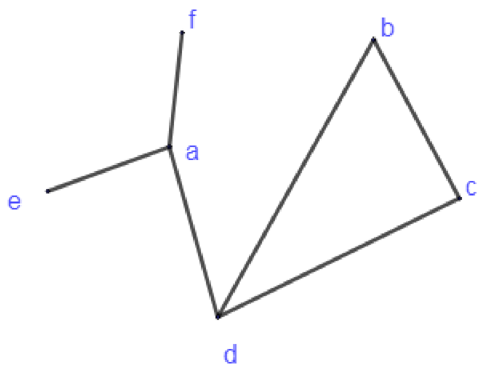

Figure 1: Graph

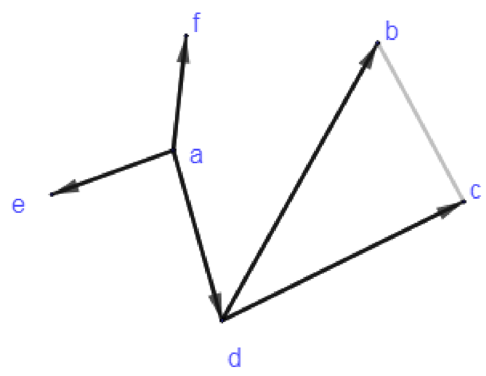

Figure 2: Digraph

Figure 1 presents a graph $G = (V(G), E(G))$ with $V(G) = \{a, b, c, d, e, f, \}$ and six edges $\{ae, af, ad, db, dc, cb\}$.

Figure 2 presents a digraph $G = (V(G), E(G))$ with $V(G) = \{a, b, c, d, e, f, \}$ and $E(G) = \{(a, e), (a, f), (a, d), (d, b), (d, c)\}$.

Definition 3.1.3 ([23, 45, 66]). Assume that $G = (V(G), E(G))$ is a digraph.

(i) G is reflexive if for each $x \in V(G)$, $(x, x) \in E(G)$.

(ii) G is transitive if for every $x, y, z \in V(G)$ with $(x, y), (y, z) \in E(G)$, we have $(x, z) \in E(G)$.

(iii) We call (V', E') a subgraph of G if $V' \subseteq V(G)$, $E' \subseteq E(G)$, and $x, y \in V'$ whenever $(x, y) \in E'$.

(iv) A (directed) walk (of length k) from x to y in a graph G is a nonempty alternating sequence $v_0e_0v_1e_1...e_{k-1}v_k$ of vertices and edges in G such that $v_0 = x$, $v_k = y$ and $e_i = (v_i, v_{i+1})$ for all $i < k$.
A directed path is a directed walk in which all vertices are distinct.

(v) For $a, b \in V(G)$, we define G-intervals along walks in the following way:

$$\begin{aligned} [a, \rightarrow)_G &= \{x \in V(G) : \text{there is a walk from } a \text{ to } x\}, \\ (\leftarrow, b]_G &= \{x \in V(G) : \text{there is a walk from } x \text{ to } b\}, \\ [a, b]_G &= [a, \rightarrow)_G \cap (\leftarrow, b]_G. \end{aligned}$$

(vi) Let A be a subset of $V(G)$. An element $b \in V(G)$ is called an upper bound of A if $a \in (\leftarrow, b]_G$ for every $a \in A$.

(vii) A subset J of $V(G)$ is directed if each finite subset of J has an upper bound in J.

We can show some G-intervals along walks of the digraph in Example 3.1.2 as follows:

$$\begin{aligned} &[a, \rightarrow)_G = \{f, e, d, b, c\}, \\ &(\leftarrow, f]_G = \{a\}, \\ &[d, \rightarrow)_G = \{b, c\}, \\ &(\leftarrow, d]_G = \{a\}. \end{aligned}$$

Note that there is also another notion of G-intervals introduced by some authors (see [8, 9, 15]) as follows:

$$\begin{aligned} [a, \rightarrow) &:= \{x \in V(G) : (a, x) \in E(G)\}, \\ (\leftarrow, b] &:= \{x \in V(G) : (x, b) \in E(G)\}, \\ [a, b] &:= [a, \rightarrow) \cap (\leftarrow, b], \end{aligned}$$

for any $a, b \in V(G)$. We can present some G-intervals of the digraph in Example 3.1.2:

$$\begin{aligned} &[a, \rightarrow) = \{f, e, d\}, \\ &(\leftarrow, f] = \{a\}, \\ &[d, \rightarrow) = \{b, c\}, \\ &(\leftarrow, d] = \{a\}. \end{aligned}$$

Note that when the digraph G is transitive, G-intervals along walks and G-intervals coincide.

Obviously, a partial order generates easily a reflexive, transitive digraph but not every digraph is generated by a partial order. Indeed, we consider the following examples.

Example 3.1.4. Let ℓ^p denote the space of all p-summable sequences of real numbers, and let $\mathcal{N}$ be a proper subset of $\mathbb{N}$. Take $V(G) = \ell^p$ and define $E(G)$ by

$$(x, y) \in E(G) \Leftrightarrow x_i \leq y_i \text{ for all } i \in \mathcal{N},$$

where $x = (x_i), y = (y_i) \in \ell^p$. Then G is a reflexive and transitive digraph.

Example 3.1.5. Let $(X, \|\cdot\|)$ be a normed space. Define a digraph G with $V(G) = X$, and

$$(x, y) \in E(G) \Leftrightarrow \|x\| \leq \|y\|$$

for $x, y \in X$. It is not difficult to show that this digraph is transitive and reflexive.

3.2. The existence of invariant G-intervals

We are now going to show the existence of a G-interval that is invariant under G-monotone mappings. First, we recall the definitions of G-monotone mappings and the finite intersection property.

Definition 3.2.1 ([66]). Let $(V(G), E(G))$ be a digraph. A mapping $T : V(G) \to V(G)$ is said to be *G*-monotone if $(T(x), T(y)) \in E(G)$ whenever $(x, y) \in E(G)$, for any $x, y \in V(G)$.

Note that if T is G-monotone, then $T(y) \in [T(x), \to)_G$ whenever $y \in [x, \to)_G$.

Example 3.2.2. Consider the digraph defined in Example 3.1.4. Fix $i_0 \in \mathcal{N}$. Define a map $T : \ell^p \to \ell^p$ by

$$T(x) = (x_1, ..., x_{i_0-1}, x_{i_0} + 1, x_{i_0+1}, ...)$$

for every $x = (x_1, ..., x_{i_0-1}, x_{i_0}, x_{i_0+1}, ...) \in \ell^p$. It is easy to check that T is G-monotone.

Definition 3.2.3. A nonempty family $\mathcal{A}$ of subsets of a set X is said to satisfy the finite intersection property if the intersection over any finite subfamily of $\mathcal{A}$ is nonempty.

Lemma 3.2.4 ([102]). *Let $(V(G), E(G))$ be a digraph. Assume that any family of G-intervals along walks in $V(G)$ having the finite intersection property has nonempty intersection. If J is a directed subset of $V(G)$, then $\bigcap_{x\in J} [x, \to)_G \neq \emptyset$.*

Proof. Take any finite subset $\{x_1, ..., x_n\}$ of J. Since J is directed, there exists a point x in J such that $x \in [x_i, \to)_G$ for every $i \in \{1, ..., n\}$, i.e., $x \in \bigcap_{i=1}^{n} [x_i, \to)_G$. By the hypothesis, we have that $\bigcap_{x\in J} [x, \to)_G \neq \emptyset$. □

The following theorem is our main result in this section.

Theorem 3.2.5 ([102]). *Let $(V(G), E(G))$ be a digraph. Assume that any family of G-intervals along walks in $V(G)$ having the finite intersection property has nonempty intersection. Let $T : V(G) \to V(G)$ be a G-monotone mapping such that $T(c) \in [c, \to)_G$ for some $c \in V(G)$. Then there exists $s \in V(G)$ such that $[s, s]_G \neq \emptyset$ and $T([s, s]_G) \subseteq [s, s]_G$.*

Proof. Set
$$I_0 = \{c, T^n(c) : n \in \mathbb{N}\}.$$
It is not difficult to prove that I_0 is a directed set and for each $x \in I_0$, $T(x) \in I_0$ and $T(x) \in [x, \rightarrow)_G$. We note that if $\mathcal{T}$ is a chain of directed subsets of $V(G)$ containing I_0 with the above properties, then $\bigcup \mathcal{T}$ is also a directed subset with the following properties: for each $x \in \bigcup \mathcal{T}$, $T(x) \in \bigcup \mathcal{T}$ and $T(x) \in [x, \rightarrow)_G$. By Kuratowski-Zorn's lemma, there exists a maximal directed set $I \subset V(G)$ which contains I_0 and satisfies above properties. It follows from Lemma 3.2.4 that the set $K := \bigcap_{x \in I}[x, \rightarrow)_G$ is nonempty. Choose finite subsets $\{x_1, ..., x_n\}$ of I and $\{y_1, ..., y_n\}$ of K. Since I is directed, $\bigcap_{i=1}^n \bigcap_{j=1}^n [x_i, y_j]_G$ is nonempty. It deduces that $K_0 := \bigcap_{x \in I, y \in K}[x, y]_G$ is nonempty. Thus there is $s \in K_0$. Clearly, for each $x \in I$, $s \in [x, \rightarrow)_G$, and hence $T(s) \in [T(x), \rightarrow)_G$. It yields $T(s) \in [x, \rightarrow)_G$ for all $x \in I$, i.e., $T(s) \in K$. Hence $T(s) \in [s, \rightarrow)_G$. Set
$$I_1 = I \cup \{s, T^n(s) : n \in \mathbb{N}\}.$$
It is not difficult to see that I_1 is a directed subset of $V(G)$ such that $I_0 \subset I_1, T(x) \in I_1$ and $T(x) \in [x, \rightarrow)_G$ for each $x \in I_1$. By the maximality of I, $I_1 = I$. It follows that both $s, T(s) \in I$. Therefore, $s \in [T(s), \rightarrow)_G$ and $T(s) \in [s, \rightarrow)_G$.

Put $H := [s, s]_G \neq \emptyset$. Take $x \in H$. Since $s \in [x, \rightarrow)_G$, we have $T(s) \in [T(x), \rightarrow)_G$ and $s \in [T(s), \rightarrow)_G$. Hence $s \in [T(x), \rightarrow)_G$, i.e., $T(x) \in (\leftarrow, s]_G$. On the other hand, $T(x) \in [T(s), \rightarrow)_G$ since $x \in [s, \rightarrow)_G$. Combining with $T(s) \in [s, \rightarrow)_G$ yields $T(x) \in [s, \rightarrow)_G$. Therefore, $T(x) \in H$ for all $x \in H$, that is, $T(H) \subseteq H$. □

Remark 3.2.6. Note that for every $a, b \in K_0$ we have $a \in [b, \rightarrow)_G$ and $b \in [a, \rightarrow)_G$. Indeed, by the above argument, $\{a, b\} \subset I \cap K$. Since $a \in I, b \in K$, we have $b \in [a, \rightarrow)_G$. And $a \in [b, \rightarrow)_G$ follows from $a \in K, b \in I$.

Furthermore, it is clear that $K_0 \subseteq [a, b]_G$. We show that $K_0 = [a, b]_G$. For this purpose, fix $t \in [a, b]_G$. Since $t \in [a, \rightarrow)_G$ and $a \in [x, \rightarrow)_G$ for each $x \in I$, we have $t \in [x, \rightarrow)_G$ for each $x \in I$. In a similar way, $t \in (\leftarrow, y]_G$ for all $y \in K$. Hence $t \in \bigcap_{x \in I, y \in K}[x, y]_G = K_0$. It implies $[a, b]_G \subseteq K_0$ and thus $K_0 = [a, b]_G$. In particular, $K_0 = [s, s]_G$ and $T(K_0) \subseteq K_0$.

We obtain the following corollary in the case of G-intervals.

Corollary 3.2.7 ([102]). *Let $(V(G), E(G))$ be a transitive digraph. Assume that any family of G-intervals in $V(G)$ having the finite intersection property has nonempty intersection. Let $T : V(G) \to V(G)$ be a G-monotone mapping such that $(c, T(c)) \in E(G)$ for some $c \in V(G)$. Then there exists $s \in V(G)$ such that $[s, s]$ is nonempty and invariant under the mapping T.*

3.3. Fixed points of monotone G-nonexpansive mappings

In this section, we establish fixed point theorems for monotone G-nonexpansive mappings.

Definition 3.3.1. Let $(X, \|\cdot\|)$ be a normed vector space endowed with a digraph $G = (V(G), E(G))$. A map $T : V(G) \to V(G)$ is said to be monotone G-nonexpansive if and only if T is G-monotone and
$$\|T(x) - T(y)\| \leq \|x - y\|$$

for all $x, y \in X$ and $y \in [x, \to)_G$.

Clearly, a G-monotone nonexpansive mapping is not necessarily continuous. Indeed, consider the following example.

Example 3.3.2. On $\mathbb{R}^2$, let

$$\|x\| := (x_1^2 + x_2^2)^{\frac{1}{2}}$$

for every $x = (x_1, x_2) \in \mathbb{R}^2$. With this norm, $(\mathbb{R}^2, \|\cdot\|)$ is a Banach space. Next, we define a digraph $G = (V(G), E(G))$ with $V(G) = \mathbb{R}^2$, and

$$\Big((x_1, x_2), (y_1, y_2)\Big) \in E(G) \Leftrightarrow x_1 = x_2 \leq y_1 = y_2$$

for $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$. The function T is defined as follows:

$$T(x_1, x_2) = \begin{cases} (1,1) & \text{if } x_1 \leq x_2 \\ (0,0) & \text{if } x_1 > x_2 \end{cases} \quad \text{for every } (x_1, x_2) \in \mathbb{R}^2.$$

Then T is monotone G-nonexpansive on $V(G)$, and it is not continuous at any point $(u, u) \in \mathbb{R}^2$.

Recently, the fixed point theory for monotone G-nonexpansive mappings has been investigated in the case of G being transitive. The initial study was carried out in Banach spaces equipped with a digraph G (see [7, 18, 66, 115]). Their results were obtained by applying iterative techniques and successive approximations.

Our approach to studying the fixed point problem for monotone G-nonexpansive mappings relies on extending established results for nonexpansive mappings. Clearly, proving any result using set-theoretical techniques in this context is challenging. To overcome this problem, it is essential to identify a subset such that a monotone G-nonexpansive mapping is nonexpansive on it. The intent behind presenting this section is to convey this significance. Subsequently, using Theorem 3.2.5 as a tool, we give some fixed-point theorems for monotone G-nonexpansive mappings in Banach spaces.

Firstly, observe that if we take $X = V(G)$ a topological Hausdorff space, and consider a partial order $\preceq := E(G)$, we obtain Theorem 1 in [47] as a corollary. In this case, $[s, s]_G = \{s\}$ reduces to a fixed point of T.

Corollary 3.3.3. *Let X be a Hausdorff topological space with a partial order $\preceq$ for which order intervals are compact, and let $T : X \to X$ be monotone. If there exists $c \in X$ such that $c \preceq T(c)$, then T has a fixed point.*

Furthermore, setting T the so-called G-regular monotone mapping of X (see [59, Definition 4]) yields Theorem 1 in [59]. But Theorem 3.2.5 is wider. From now on, we use the notation $(X, \|\cdot\|)$ to represent a Banach space equipped with a digraph $G = (V(G), E(G))$, and C is a subset of X such that $C = V(G)$.

Recall that a Banach space X has the fixed point property for nonexpansive mappings (FPP, for short) if every nonexpansive (i.e., 1-Lipschitz) mapping $T : C \to C$ acting on a bounded closed convex subset C of X has a fixed point. There is an extensive literature on metric fixed point theory originated in the 1965 existence theorems of Browder, Göhde and Kirk, see [54, 72, 113] for more details. In 2006, García Falset, Lloréns Fuster and Mazcuñan Navarro [48] showed that if X is a reflexive Banach space, and $\Gamma'_X(0) := \lim_{t\to 0^+} \frac{\Gamma_X(t)}{t} < 1$, where $\Gamma_X = \sup\big\{ \inf_{n>1} \big(\frac{1}{2}(\|x_1 + tx_n\| + \|x_1 - tx_n\|) - 1\big) : (x_n) \subset B(0,1), x_n \rightharpoonup 0\big\}$ is the modulus of nearly uniform smoothness of X, then X has FPP. In particular, uniformly nonsquare Banach spaces have FPP (recall that a Banach space is uniformly nonsquare if $\sup_{x,y\in S_X} \min\{\|x+y\|, \|x-y\|\} < 2$). However, it is well known that there exist Banach spaces, for example c_0, for which not every bounded closed convex subset has the fixed point property. This fact prompted many researchers to study fixed points of nonexpansive mapping under the stronger assumption of weak compactness of C. A Banach space X is said to have the weak fixed point property for nonexpansive mappings (WFPP, for short) if every nonexpansive mapping $T : C \to C$ acting on a convex and weakly compact subset C of X has a fixed point. The weak topology can be replaced by some other topologies such as the weak* topology or the topology of convergence in measure. In the theorem below we restrict ourselves to the weak topology.

Theorem 3.3.4 ([102]). *Let $(X, \|\cdot\|)$ be a Banach space with a digraph G, and let C be a weakly compact convex subset of X such that $C = V(G)$. Assume that G-intervals along walks are closed and convex, and X has WFPP. If $T : C \to C$ is a monotone G-nonexpansive mapping, and there exists $c \in V(G)$ such that $T(c) \in [c, \to)_G$, then there exists $x_0 \in C$ such that $T(x_0) = x_0$.*

Proof. Since C is weakly compact and G-intervals along walks are convex and closed, any family of such G-intervals in C having the finite intersection property has nonempty intersection. Now, it follows from Theorem 3.2.5 that there exists $s \in C$ such that $[s, s]_G$ is a nonempty convex and weakly compact subset of C. If $x, y \in [s, s]_G$, then $x \in (\leftarrow, s]_G, y \in [s, \to)_G$, and consequently $y \in [x, \to)_G$. Thus each monotone G-nonexpansive mapping is nonexpansive on $[s, s]_G$. Since X has WFPP, there exists $x_0 \in C$ such that $T(x_0) = x_0$. □

In particular, Theorem 3.3.4 holds if X is uniformly nonsquare, uniformly noncreasy (see [98]) or has the weak normal structure.

Remark 3.3.5. The assumption that G-intervals along walks are convex appears to be strong. However, if the digraph G is transitive, then G-intervals along walks coincide with the "usual" G-intervals as defined in Section 3.1. Note that transitivity of digraphs is a common assumption in this branch of fixed point theory (compare [11, 13]).

Remark 3.3.6. Recall that a subset C of a Banach space X has *the hereditary fixed point property for nonexpansive mappings* (HFPP, for short) if every nonexpansive mapping $T : C \to C$ has a fixed point in any nonempty bounded closed and convex subset of C that is invariant under T. Notice that Theorem 3.3.4 remains true if we replace the assumption "X has WFPP" by "C has HFPP".

3.4. Fixed points of monotone G-asymptotically nonexpansive mappings

The second interesting class of mappings that finds its root in the work of Goebel and Kirk [52] are asymptotic nonexpansive mappings. We are going to extend our results to this case.

Definition 3.4.1. Let $(X, \|\cdot\|)$ be a normed vector space endowed with a digraph G and let C be a subset of X. A mapping $T : C \to C$ is said to be monotone G-asymptotically nonexpansive if T is G-monotone and there exists a sequence of positive numbers $\{k_n\}$ such that $\lim_{n\to\infty} k_n = 1$ and

$$\|T^n(x) - T^n(y)\| \leq k_n\|x - y\|$$

for any $n \in \mathbb{N}$, and any $x, y \in C$ such that $y \in [x, \to)_G$.

As in the case of nonexpansive mappings, a Banach space X has the weak fixed point property for asymptotically nonexpansive mappings if every asymptotically nonexpansive mapping $T : C \to C$ acting on a weakly compact and convex subset C of X has a fixed point. The following theorem is a rather obvious counterpart of Theorem 3.3.4.

Theorem 3.4.2 ([102]). *Let $(X, \|\cdot\|)$ be a Banach space with a digraph G, and let C be a weakly compact convex subset of X such that $C = V(G)$. Assume that G-intervals along walks are convex and closed. Let $T : C \to C$ be a monotone G-asymptotically nonexpansive mapping. If X has the weak fixed point property for asymptotically nonexpansive mappings, and there exists $c \in V(G)$ such that $T(c) \in [c, \to)_G$, then T has a fixed point in C.*

Proof. Since C is weakly compact and G-intervals along walks are convex and closed, any family of such G-intervals in C having the finite intersection property has nonempty intersection. Theorem 3.2.5 shows that there exists $s \in C$ such that $[s, s]_G$ is a nonempty convex and weakly compact subset of C and $T([s, s]_G) \subseteq [s, s]_G$. Since T is asymptotically nonexpansive on $[s, s]_G$ and X has the weak fixed point property for asymptotically nonexpansive mappings, there exists $x_0 \in C$ such that $T(x_0) = x_0$. □

Fixed point theory for asymptotically nonexpansive mappings is worse understood than its nonexpansivity counterpart. The original result from [52] stating that asymptotically nonexpansive mappings have the fixed point property in each closed convex and bounded subset of a uniformly convex space was generalized in [118] when X is nearly uniformly convex, in [86] when X satisfies the uniform Opial condition, and in [76] when X has uniform normal structure. In 2012, it was proved in [116] that the super fixed point property of X for nonexpansive mappings is equivalent to the super fixed point property for asymptotically nonexpansive mappings (a Banach space X has the super fixed point property for nonexpansive mappings if every Banach space Y which is finitely representable in X has FPP). In particular, Theorem 3.4.2 is true if X is uniformly nonsquare, uniformly noncreasy or any ψ-direct sum $X_1 \oplus_\psi X_2$ of uniformly nonsquare Banach spaces (see [117, Theorem 3.7]).

Remark 3.4.3. Even in the case of a uniformly convex space Theorem 3.4.2 is stronger that the corresponding result in [10], where G is a partial order and $T : C \to C$ is a continuous monotone G-asymptotically nonexpansive mapping.

3.5. Common fixed point for a commutative family of monotone mappings

In 2018, Espínola and Wiśnicki [47] proved the existence of a common fixed point for a commutative family of monotone mappings of a Hausdorff topological space with a partial order. In this section, we generalize it and obtain an invariance theorem for commutative families of G-monotone mappings on a set equipped with a digraph.

Theorem 3.5.1 ([102]). *Let $(V(G), E(G))$ be a digraph, and $\mathcal{F}$ be a commutative family of G-monotone mappings from $V(G)$ into itself. Assume that any family of G-intervals along walks in $V(G)$ having the finite intersection property has nonempty intersection. Moreover, there exists $c \in V(G)$ such that $T(c) \in [c, \to)_G$ for every $T \in \mathcal{F}$. Then there is $s \in V(G)$ such that $[s, s]_G \neq \emptyset$ and $T([s, s]_G) \subseteq [s, s]_G$ for every $T \in \mathcal{F}$.*

Proof. Let

$$\mathcal{F}_1 = \{T_1 \circ T_2 \circ ... \circ T_n : T_i \in \mathcal{F}, i = 1, ..., n,\ n \in \mathbb{N}\}, \quad L_0 = \{c, T(c) : T \in \mathcal{F}_1\}.$$

By commutativity and monotonicity of $\mathcal{F}$, it is not difficult to see that for each $T \in \mathcal{F}_1$, T is G-monotone, and L_0 is a directed set. Moreover, $T(x) \in L_0$ and $T(x) \in [x, \to)_G$ for every $T \in \mathcal{F}_1$, $x \in L_0$. By an argument analogous to that used in the proof of Theorem 3.2.5, there exists $s \in V(G)$ such that $[s, s]_G$ is nonempty and $T([s, s]_G) \subseteq [s, s]_G$ for every $T \in \mathcal{F}_1$. Since $\mathcal{F} \subseteq \mathcal{F}_1$, the proof is complete. □

Corollary 3.5.2 ([102]). *Let $(V(G), E(G))$ be a transitive digraph, and $\mathcal{F}$ be a commutative family of G-monotone mappings from $V(G)$ into itself. Assume that any family of G-intervals in $V(G)$ having the finite intersection property has nonempty intersection. Furthermore, suppose that there exists $c \in V(G)$ such that $(c, T(c)) \in E(G)$ for $T \in \mathcal{F}$. Then there is $s \in V(G)$ such that $[s, s]_G$ is nonempty and invariant under every $T \in \mathcal{F}$.*

In 1974, Bruck [30] proved that any commutative family of nonexpansive self-mapping of a closed convex subset C of a Banach space has a common fixed point if C is weakly compact or bounded and separable provided that C has the HFPP for nonexpansive mappings. The following theorem is a counterpart of Theorem 3.3.4.

Theorem 3.5.3 ([102]). *Let $(X, \|\cdot\|)$ be a Banach space with a digraph G, C a weakly compact convex subset of X such that $C = V(G)$, and $\mathcal{F}$ a nonempty commutative family of monotone G-nonexpansive mappings from C into C. Assume that G-intervals along walks are convex and closed. If X has the WFPP for nonexpansive mappings, and there exists $c \in V(G)$ such that $T(c) \in [c, \to)_G$ for every $T \in \mathcal{F}$, then there is $x_0 \in C$ such that $T(x_0) = x_0$ for every $T \in \mathcal{F}$.*

Proof. Since C is weakly compact and G-intervals along walks are convex and closed, any family of such G-intervals in C having the finite intersection property has nonempty intersection. Theorem 3.5.1 now shows that there exists $s \in C$ such that $[s, s]_G$ is a nonempty convex and weakly compact subset of C, and $T([s, s]_G) \subseteq [s, s]_G$ for any $T \in \mathcal{F}$. Since T is nonexpansive on $[s, s]_G$ and X has WFPP (and hence C has HFPP), it follows from Theorem 1 in [30] that there exists $x_0 \in C$ such that $T(x_0) = x_0$ for every $T \in \mathcal{F}$. □

In particular, Theorem 3.5.3 holds if X is uniformly nonsquare, uniformly noncreasy or has weak normal structure.

There exists relatively few results concerning the existence of common fixed points for commutative families of asymptotically nonexpansive mappings. Yet, it was proved in [116, Theorem 3.3] that if a Banach space X has the super fixed point property for nonexpansive mappings, then any commutative family of asymptotically nonexpansive mappings acting on a closed convex and bounded subset of X has a common fixed point. Combining it with Theorem 3.5.1 yields

Theorem 3.5.4 ([102]). *Let $(X, \|.\|)$ be a Banach space with a digraph G, C a bounded closed convex subset of X such that $C = V(G)$, and $\mathcal{F}$ a nonempty commutative family of monotone G-asymptotically nonexpansive mappings from C into C. Assume that G-intervals along walks are convex and closed. If X has the super fixed point property for nonexpansive mappings, and there exists $c \in V(G)$ such that $T(c) \in [c, \rightarrow)_G$ for every $T \in \mathcal{F}$, then there is $x_0 \in C$ such that $T(x_0) = x_0$ for every $T \in \mathcal{F}$.*

Proof. Note that X is superreflexive since it has the super fixed point property for nonexpansive mappings. Hence C is weakly compact and so that any family of G-intervals along walks in C having the finite intersection property has nonempty intersection. Furthermore, there exists $s \in C$ such that $[s, s]_G$ is a nonempty convex and weakly compact subset of C that is invariant under $\mathcal{F}$. Since X has the super fixed point property for nonexpansive mappings, it follows from Theorem 3.3 in [116] that there exists $x_0 \in C$ such that $T(x_0) = x_0$ for every $T \in \mathcal{F}$. □

3.6. Application

Let $X = L^2([0, 1], \mathbb{R})$ be the space of measurable functions $x : [0, 1] \to \mathbb{R}$ such that

$$\int_0^1 x^2(t)dt < \infty.$$

Note that X is a Hilbert space with the norm

$$\|x\| = \Big(\int_0^1 x^2(t)dt\Big)^{1/2}.$$

Fix $\alpha \in (0, 1)$, and consider a digraph G on X as follows:

$$(x, y) \in E(G) \Leftrightarrow x(s) \leq y(s) \text{ almost everywhere in } [0, \alpha],$$

where $x, y \in X$. Clearly, the digraph is reflexive, transitive and G-intervals are convex. We are going to show that G-intervals are also closed. For this purpose, we take G-interval of the form $[a, \to) = \{x \in X : (a, x) \in E(G)\}$, for $(\leftarrow, a] = \{x \in X : (x, a) \in E(G)\}$ we proceed analogously. Assume that $(x_n)_n$ is a sequence in $[a, \to)$ such that $x_n \to x$ as $n \to \infty$. We prove that $x \in [a, \to)$, i.e. $a(t) \le x(t)$ for a.e. on $[0, \alpha]$. For each $n \in \mathbb{N}$, let $J_n = \{t \in [0, \alpha] : a(t) - x(t) > \dfrac{1}{n}\}$. We have

$$0 \le \lim_{m \to \infty} \int_{J_n} (x_m(t) - a(t))dt = \int_{J_n} (x(t) - a(t))dt \le -\frac{1}{n}\mu(J_n) \le 0,$$

and hence $\mu(J_n) = 0$. Thus $\mu(\bigcup_{n=1}^{\infty} J_n) = 0$, implying that $x(t) - a(t) \ge 0$ a.e. on $[0, \alpha]$.

Put $\Delta := \{(t, s) : 0 \le s \le t \le 1\}$. Suppose that $g \in X$, $F : \Delta \times X \to \mathbb{R}$ is measurable in its first coordinate and the following conditions are satisfied:

(i) For almost every $(t, s) \in \Delta_\alpha := \{(t, s) : 0 \le s \le t \le \alpha\}$, $x, y \in X$ such that $(x, y) \in E(G)$, we have $F(t, s, x) \le F(t, s, y)$;

(ii) For almost every $(t, s) \in \Delta$, $x, y \in X$, $(x, y) \in E(G)$, we have $|F(t, s, x) - F(t, s, y)| \le |x(s) - y(s)|$;

(iii) There exists a non-negative function $h(\cdot, \cdot) \in L^2([0, 1] \times [0, 1])$ and $\beta \in [0, \frac{1}{2})$ such that $|F(t, s, x)| \le h(t, s) + \beta|x(s)|$ for almost every $(t, s) \in \Delta$ and $x \in X$.

Define

$$Tx(t) = g(t) + \int_0^t F(t, s, x)ds, \quad t \in [0, 1], \tag{3.6.1}$$

and suppose that $t \in [0, 1]$, $t \mapsto \int_0^t F(t, s, x)ds$ is measurable for each $x \in X$. Clearly, T is an operator from $L^2([0, 1], \mathbb{R})$ to itself. We have

$$\begin{aligned}
\|T(x)\|^2 &= \int_0^1 \left|(g(t) + \int_0^t F(t, s, x)ds\right|^2 dt \\
&\le 2\int_0^1 |g(t)|^2 dt + 2\int_0^1 \int_0^t |F(t, s, x)|^2 dsdt \\
&\le 2\int_0^1 |g(t)|^2 dt + 2\int_0^1 \int_0^t |h(t, s) + \beta|x(s)||^2 dsdt \\
&\le 2\int_0^1 |g(t)|^2 dt + 2\int_0^1 \int_0^1 |h(t, s) + \beta|x(s)||^2 dsdt \\
&\le 2\int_0^1 |g(t)|^2 dt + 4\int_0^1 \int_0^1 |h(t, s)|^2 dsdt + 4\beta^2 \int_0^1 \int_0^1 |x(s)|^2 dsdt \\
&\le 2\int_0^1 |g(t)|^2 dt + 4\int_0^1 \int_0^1 |h(t, s)|^2 dsdt + 4\beta^2 \|x\|^2
\end{aligned}$$

We choose $r > 0$ such that

$$r^2 \ge \frac{1}{1 - 4\beta^2}\Big(2\int_0^1 |g(t)|^2 dt + 4\int_0^1 \int_0^1 |h(t, s)|^2 dsdt\Big).$$

Thus we have that if $\|x\| \le r$ then $\|T(x)\| \le r$. Therefore $T(\overline{B}_r(0)) \subseteq \overline{B}_r(0)$.

By condition (i), it is easy to see that T is G-monotone. For $x, y \in X$ such that $(x, y) \in E(G)$, we have

$$\begin{aligned}\|Tx - Ty\|^2 &= \int_0^1 (Tx(t) - Ty(t))^2 dt = \int_0^1 \Big(\int_0^t (F(t,s,x) - F(t,s,y)ds \Big)^2 dt \\ &\le \int_0^1 \Big(\int_0^t |x(s) - y(s)|ds \Big)^2 dt \le \int_0^1 \Big(\int_0^1 |x(s) - y(s)|ds \Big)^2 dt \\ &= \int_0^1 \|x - y\|^2 dt = \|x - y\|^2.\end{aligned}$$

This implies that T is a monotone G-nonexpansive operator. From these, we get the following result which is the mixture of fixed point theorems for nonexpansive and monotone mappings. Notice that in condition (i) we assume that $F(t,s,x) \le F(t,s,y)$ for almost every $(t,s) \in \Delta_\alpha$ only, and thus F does not need to be monotone on the whole Δ.

Theorem 3.6.1 ([102]). *Assume that the above conditions (i)-(iii) are satisfied. If $g(t) + \int_0^t F(t,s,0)ds \ge 0$ for almost every $t \in [0,\alpha]$, then the Volterra type integral equation*

$$x(t) = g(t) + \int_0^t F(t,s,x)dt, \quad t \in [0,1],$$

has a solution in $L^2([0,1],\mathbb{R})$ that is non-negative almost everywhere on $[0,\alpha]$.

Proof. We consider the operator T defined by (3.6.1) on the closed ball $\overline{B}_r(0)$ with the radius r large enough so that $T(\overline{B}_r(0)) \subseteq \overline{B}_r(0)$. Put

$$\mathcal{L} = \{\overline{B}_r(0) \cap I : I \text{ is a } G\text{-interval in } L^2([0,1],\mathbb{R})\}.$$

Since $\overline{B}_r(0)$ is weakly compact in $L^2([0,1],\mathbb{R})$, any subfamily of $\mathcal{L}$ having the finite intersection property has nonempty intersection. The condition $g(t) + \int_0^t F(t,s,0)ds \ge 0$ implies $T(0)(t) \ge 0$ for almost every $t \in [0,\alpha]$, i.e., $(0, T(0)) \in E(G)$. It is well known that $L^2([0,1],\mathbb{R})$ has the fixed point property for nonexpansive mappings. Therefore, the conclusion follows from Theorem 3.3.4. $\square$

Chapter 4

Fixed points of G-monotone mappings in geodesic spaces

In the results of Chapter 3, the assumption *"Any family of G-intervals in $V(G)$ of a digraph $G = (V(G), E(G))$ having the finite intersection property has a nonempty intersection"* plays an important role in our arguments. In this chapter, we are going to show that $V(G)$ admits the above assumption for any family of nonempty bounded closed convex G-intervals if we consider $V(G)$ as a uniformly convex geodesic space X. Next, we establish the results regarding the existence of fixed points for monotone G-nonexpansive mappings and monotone G-nonexpansive multivalued mappings in geodesic spaces with a digraph $G = (V(G), E(G))$. First, we recall some basic notions about geodesic spaces and uniform convexity.

4.1. Preliminaries

Let (X, d) be a metric space. The space X is said to admit the midpoints if for every $x, y \in X$, there exists a point denoted by $m(x, y) \in X$, called a midpoint of x and y, satisfying

$$d(x, m(x, y))) = d(y, m(x, y)) = \frac{1}{2}d(x, y).$$

A geodesic path (or simply a geodesic) in X is a path $\gamma : [0, l] \subseteq \mathbb{R} \to X$ such that γ is an isometry. If γ is a geodesic path such that $\gamma(0) = x$ and $\gamma(l) = y$, we say that γ joins x and y, and the image of γ is called a geodesic segment from x to y, denoted by $[x, y]$.

The space X is a (uniquely) geodesic space if any two points in X are joined by a (uniquely) geodesic path. It is well-known that any complete metric space which admits midpoints is a geodesic space. We refer to [27] for details on geodesic spaces.

Let X be a uniquely geodesic space. A point $z \in X$ belongs to the unique geodesic segment $[x, y]$ if and only if there exists a unique $\alpha \in [0, 1]$ such that

$$d(x, z) = (1 - \alpha)d(x, y) \quad \text{and} \quad d(z, y) = \alpha d(x, y),$$

and we write $z = \alpha x \oplus (1 - \alpha)y$. For $\alpha = 1/2$ we get the midpoint $m(x, y)$ of x and y.

A set $C \subset X$ is convex if geodesic segments $[x, y] \subset C$ for any $x, y \in C$.

Example 4.1.1. 1) The metric spaces $(\mathbb{R}^k, d)$, $(\mathbb{R}^k, d_1)$, $(\mathbb{R}^k, d_2)$ presented in Example 2.1.9 are uniquely geodesic spaces.

2) Examples of nonlinear uniquely geodesic spaces include Hadamard manifolds [31] and CAT(0) spaces [70].

Let us discuss some notions regarding the convexity of the metric d. The simplest form of this metric convexity requires balls to be convex, or the mapping $x \mapsto d(x, y)$ to be convex for every fixed $y \in X$. It is not difficult to see that both conditions are equivalent in normed spaces with strictly convex norms (in meaning that if $x \neq y$ and $\|x\| = \|y\| = 1$, we have $\|x + y\| < 2$). However, Busemann and Phadke [32] showed that there exist metric spaces whose balls are convex but its metric is not. After that, Foertsch [49] proved that in Busemann spaces ([19, Definition 1.1.4]) both concepts are equivalent. Recall that a Busemann space (X, d) (also known as a hyperbolic metric space, see [104]) is a uniquely geodesic space (X, d) such that

$$d(tx_1 \oplus (1-t)x_2, ty_1 \oplus (1-t)y_2) \leq td(x_1, y_1) + (1-t)d(x_2, y_2),$$

for all x_1, x_2, y_1, y_2 in X and $t \in [0, 1]$.

The study of a stronger form of convexity in Banach spaces has appeared for a long time. The notion of uniform convexity for Banach spaces was defined by Clarkson [35] as follows: a Banach space $(X, \|\cdot\|)$ is said to be uniformly convex if for every $\varepsilon \in (0, 2]$, there exists $\delta(\varepsilon) > 0$ such that for any two points $x, y \in X$ with $\|x\| = \|y\| = 1$, if $\|x - y\| \geq \varepsilon$, then $\|\frac{x+y}{2}\| \leq 1 - \delta(\varepsilon)$.

After that, there have been many efforts to define uniformly convex structures on non-linear spaces. In 2008, Gelander, Karlsson and Margulis [51] considered strictly convex geodesic spaces as follows: a uniquely geodesic space (X, d) is said to be strictly convex if $d(a, \frac{1}{2}x \oplus \frac{1}{2}y) < \max\{d(a, x), d(a, y)\}$ for every $a, x, y \in X$ with $d(x, y) > 0$. Then X is called weakly uniformly convex if for any $a \in X$, the modulus of convexity

$$\delta(a, r, \varepsilon) = \inf\left\{r - d(a, \frac{1}{2}x \oplus \frac{1}{2}y) : d(a, x) \leq r, d(a, y) \leq r, d(x, y) \geq \varepsilon r\right\} > 0$$

for any $\varepsilon > 0$, $r > 0$. They also called X uniformly convex if $\forall \varepsilon > 0, \exists \eta(\varepsilon) > 0$ such that $\forall r > 0$, $x \in X$, $\delta(x, r, \varepsilon) \geq \eta(\varepsilon) r$.

In [63], Kell stated that a metric space X admitting midpoints is uniformly p-convex if for every $\varepsilon > 0$ there exists $\rho_p(\varepsilon) \in (0, 1)$ such that for all $x, y, a \in X$ satisfying $d(x, y) > \varepsilon \mathcal{M}^p(d(x, a), d(y, a))$ for $p > 1$, and $d(x, y) > |d(x, a) - d(y, a)| + \varepsilon \mathcal{M}^1(d(x, a), d(y, a))$ for $p = 1$, we have

$$d(a, m(x, y)) \leq (1 - \rho_p(\varepsilon))\mathcal{M}^p(d(x, a), d(y, a)),$$

where $\mathcal{M}^p(t, u) = \left(t^p/2 + u^p/2\right)^{1/p}$, and $\mathcal{M}^\infty(t, u) = \max\{t, u\}$. He also proved that any uniformly p-convex space ($p \geq 1$) is uniformly ∞-convex. It is not difficult to see that any uniformly ∞-convex uniquely geodesic space is uniformly convex in the sense of Gelander–Karlsson–Margulis.

With $p \in (1, \infty)$, p-uniformly convex geodesic spaces with parameter $k > 0$, as defined by Naor and Silberman in [95], are also special cases of the two definitions mentioned above. Recently, Kuwae [81], based on the approach of Naor and Silberman, studied spaces with p-uniform convexity similar to that of Banach spaces.

In 2015, Leuştean and Nicolae [83] defined weak uniform convexity in an alternative way: A geodesic space (X, d) is weakly uniformly convex if there exists a mapping $\delta' : X \times (0, \infty) \times (0, 2] \to (0, 1]$ such that for any $a \in X$, $r > 0$, $\varepsilon \in (0, 2]$, for every $x, y \in X$, if $d(a, x) \leq r$, $d(a, y) \leq r$, and $d(x, y) > 0$ then $d(a, m(x, y)) \leq (1 - \delta'(a, r, \varepsilon))r$. Such a mapping δ' is referred to as a modulus of weak uniform convexity. In their paper, they also assumed that for all $a \in X$, $\varepsilon \in (0, 2]$, there exists $s > 0$ such that $\inf_{r \geq s} \delta'(a, r, \varepsilon) > 0$. We can easily prove that any uniformly ∞-convex space is weakly uniformly convex in the sense of Leuştean–Nicolae. Moreover, in strictly convex uniquely geodesic space (X, d), if X is weakly uniformly convex in the sense of Gelander–Karlsson–Margulis, then X is weakly uniformly convex in the sense of Leuştean–Nicolae. Indeed, we can choose the function $\delta'(\cdot, \cdot, \cdot)$ defined by $\delta'(a, r, \varepsilon) := \inf\{r - d(a, \frac{1}{2}x \oplus \frac{1}{2}y) : x, y \in X, d(a, x) \leq r, d(a, y) \leq r, d(x, y) \geq \varepsilon r\}$. This function is also used in the research conducted by Dehaish and Khamsi [37] while investigating the existence of fixed points for monotone nonexpansive mappings in hyperbolic metric spaces.

Recently, Quan [99] introduced p-uniformly convex structure for a hyperbolic metric space as follows: assume that (X, d) is a hyperbolic metric space, $p \geq 2$. For each $a \in X$, $r > 0$ and $\varepsilon \geq 0$, we set

$$\psi(a, r, \varepsilon) = \inf\Big\{\frac{1}{2}d^p(a, x) + \frac{1}{2}d^p(a, y) - d^p\Big(a, \frac{1}{2}x \oplus \frac{1}{2}y\Big)\Big\},$$

where the infimum is taken over all $x, y \in X$ such that $d(a, x) \leq r$, $d(a, y) \leq r$ and $d(x, y) \geq r\varepsilon$. We say that (X, d) is p-uniformly convex (p-UC for short) if

$$c_X = \inf\Big\{\frac{\psi(a, r, \varepsilon)}{r^p \varepsilon^p} : a \in X, r > 0, \varepsilon > 0\Big\} > 0.$$

Clearly, any hyperbolic metric space with p-uniform convexity is weakly uniformly convex in the sense of Leuştean-Nicolae. Indeed, all we need is to choose the function $\delta'(a, r, \varepsilon) := c_X \varepsilon^p / p$.

In this chapter, we drop the assumption about hyperbolicity of the space and define uniform convexity in the same way as Dehaish and Khamsi did.

Definition 4.1.2. Let (X, d) be a uniquely geodesic metric space. For any $a \in X$, $r > 0$ and $\varepsilon > 0$, define

$$D_a(r, \varepsilon) = \{(x, y) \in X \times X : d(x, a) \leq r, d(y, a) \leq r, d(x, y) \geq r\varepsilon\},$$

and let

$$\delta(a, r, \varepsilon) = \inf\Big\{1 - \frac{1}{r}d\Big(\frac{1}{2}x \oplus \frac{1}{2}y, a\Big) : (x, y) \in D_a(r, \varepsilon)\Big\}.$$

In the above, we adopt the convention that $\inf \emptyset = 1$.

(i) We say that X is weakly uniformly convex (WUC for short) if $\delta(a, r, \varepsilon) > 0$ for any $a \in X$, $r > 0$, and $\varepsilon > 0$.

(ii) We say that X is uniformly convex (UC for short) if for every $s > 0$ and $\varepsilon > 0$, there exists $\eta(s, \varepsilon) > 0$ such that $\delta(a, r, \varepsilon) > \eta(s, \varepsilon) > 0$ for any $a \in X$, $r > s$.

Let us recall and prove some properties of uniformly convex geodesic metric spaces. Most of these results were presented by us in [99], [101].

Lemma 4.1.3. *Let (X,d) be a uniquely geodesic space. Let $a \in X$, $r > 0$, and $\varepsilon \geq 0$.*

(a) $\delta(a,r,0) = 0$, and $\delta(a,r,\cdot)$ is an increasing function on $[0,2)$;

(b) Assume that $r_2 \geq r_1 > 0$. Then

$$1 - \frac{r_2}{r_1}(1 - \delta(a, r_2, \varepsilon\frac{r_1}{r_2})) \leq \delta(a, r_1, \varepsilon);$$

(c) Suppose that X is WUC and $t_n > 0$ for all $n \geq 1$. If $\lim\limits_{n\to\infty} \delta(a,r,t_n) = 0$ for a fixed $a \in X$ and $r > 0$, then $\inf\limits_{n\geq 1} t_n = 0$.

Proof. It is not difficult to show (a) and (b). We are going to prove (c). Assume that $\lim\limits_{n\to\infty} \delta(a,r,t_n) = 0$ and $\inf\limits_{n\geq 1} t_n \neq 0$. Then there exists α such that

$$0 < \alpha \leq \inf_{n\geq 1} t_n.$$

Consequently, $\alpha \leq t_n$ for all $n \geq 1$. Since the function $\delta(a,r,\varepsilon)$ is increasing of ε, we have

$$\delta(a,r,\alpha) \leq \delta(a,r,t_n), \tag{4.1.1}$$

for every $n \geq 1$. Taking the limit on both sides of (4.1.1) as $n \to \infty$, we have

$$0 < \delta(a,r,\alpha) \leq \lim_{n\to\infty} \delta(a,r,t_n).$$

It contradicts $\lim\limits_{n\to\infty} \delta(a,r,t_n) = 0$. Therefore, $\inf\limits_{n\geq 1} t_n = 0$. □

Notice that if a uniquely geodesic metric space X is uniformly convex, then all balls are convex. In fact, we have a stronger conclusion.

Lemma 4.1.4. *Let (X,d) be a complete uniquely geodesic metric space. Let $r > 0$, $a \in X$.*

i) Assume that X is WUC. Let $t \in [\alpha,\beta]$, where $0 < \alpha \leq \beta < 1$. If

$$d(x,a) \leq r, d(y,a) \leq r, d(x,y) \geq r\varepsilon$$

for some $\varepsilon > 0$, $x,y \in X$, then there exists $\delta(a,r,2\varepsilon\min\{\alpha,1-\beta\}) \in (0,1)$ such that

$$d(a,(1-t)x \oplus ty) \leq r\Big(1 - \delta(a,r,2\varepsilon\min\{\alpha,1-\beta\})\Big).$$

ii) Assume that $t_n \in [\alpha,\beta]$ for every $n \geq 1$, where $0 < \alpha \leq \beta < 1$, and $(x_n)_n$, $(y_n)_n$ are two sequences in X such that $\limsup\limits_{n\to\infty} d(a,x_n) \leq r$, $\limsup\limits_{n\to\infty} d(a,y_n) \leq r$, and $\lim\limits_{n\to\infty} d\Big(a, t_nx_n \oplus (1-t_n)y_n\Big) = r$. If X is UC, then $\lim\limits_{n\to\infty} d(x_n,y_n) = 0$.

Proof. (i) Take $\varepsilon > 0$, $x, y \in X$. Without loss of generality we may assume that $t < 1/2$. Let $z_t = (1-t)x \oplus ty$ and $z_{2t} = (1-2t)x \oplus 2ty$ so that

$$d(x, z_t) = td(x, y), \quad d(y, z_t) = (1-t)d(x, y),$$

and

$$d(x, z_{2t}) = 2td(x, y), \quad d(y, z_{2t}) = (1-2t)d(x, y).$$

Note that $d(z_t, z_{2t}) = td(x, y)$. Hence $d(x, z_t) = d(z_t, z_{2t}) = td(x, y) = 1/2d(x, z_{2t})$. It implies $z_t = \frac{1}{2}x \oplus \frac{1}{2}z_{2t}$. Since $t \geq \min\{\alpha, 1-\beta\}$, we have

$$d(x, z_{2t}) = 2td(x, y) \geq 2r\varepsilon \min\{\alpha, 1-\beta\}.$$

Since X is WUC, there exists $\delta(a, r, 2\varepsilon \min\{\alpha, 1-\beta\}) > 0$

$$d(a, z_t) \leq r(1 - \delta(a, r, 2\varepsilon \min\{\alpha, 1-\beta\})).$$

(ii) For each $n \geq 1$, define

$$r_n = \max\{d(a, x_n), d(a, y_n)\}.$$

Hence

$$\begin{aligned}\limsup_{n\to\infty} r_n &= \limsup_{n\to\infty} \max\{d(a, x_n), d(a, y_n)\} \\ &= \max\{\limsup_{n\to\infty} d(a, x_n), \limsup_{n\to\infty} d(a, y_n)\} \leq r.\end{aligned}$$

We note that the sequences $(d(a, x_n))_n$ and $(d(a, y_n))_n$ are bounded so that there exists $R > 0$ such that $r_n \leq R$ for all $n \geq 1$.

Case 1. If $\limsup_{n\to\infty} r_n = 0$, then $\limsup_{n\to\infty} d(a, x_n) = \limsup_{n\to\infty} d(a, y_n) = 0$. It deduces that $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Case 2. Let $d = \limsup_{n\to\infty} r_n > 0$. Without loss of generality, we assume that $\lim_{n\to\infty} d(x_n, y_n) \neq 0$. Then there exists $\varepsilon > 0$ and subsequences $(x_{n_k})_k$, $(y_{n_k})_k, (r_{n_k})_k$ such that

$$d(x_{n_k}, y_{n_k}) \geq \varepsilon \ \text{ and } \ r_{n_k} > d - \varepsilon > 0$$

for any $k \geq k_0$. We have

$$d(x_{n_k}, y_{n_k}) \geq \varepsilon \geq r_{n_k}\frac{\varepsilon}{R}.$$

Since X is UC and (i) holds, we have that there exists $\eta\Big(d-\varepsilon, 2\min\{\alpha, 1-\beta\}\frac{\varepsilon}{R})\Big) \in (0, 1)$ such that

$$\begin{aligned}d\Big(a, t_{n_k}x_{n_k} \oplus (1-t_{n_k})y_{n_k}\Big) &\leq r_{n_k}\Big(1 - \delta(a, r_{n_k}, 2\min\{\alpha, 1-\beta\}\frac{\varepsilon}{R})\Big) \\ &< r_{n_k}\Big(1 - \eta(d-\varepsilon, 2\min\{\alpha, 1-\beta\}\frac{\varepsilon}{R})\Big)\end{aligned}$$

for any $k \geq k_0$. Taking limsup as $k \to \infty$, we get

$$r \leq d\Big(1 - \eta(d-\varepsilon, 2\min\{\alpha, 1-\beta\}\frac{\varepsilon}{R})\Big) < r,$$

which is the desired contradiction. Therefore, $\lim_{n\to\infty} d(x_n, y_n) = 0$. □

The following theorem has been proved in [70] in the case of hyperbolic uniformly convex spaces.

Theorem 4.1.5. *Let (X, d) be a complete uniquely geodesic metric space, C a nonempty closed convex subset of X, and $a \in X$. Assume that X is UC. Let $d(a, C) = \inf\{d(a, y) : y \in C\}$. Then there exists a unique $c \in C$ such that $d(a, C) = d(a, c)$.*

Proof. If $d(a, C) = 0$, then there exists a sequence $(x_n)_n$ of elements of C that tends to a, and since C is closed, $c := a \in C$. Thus we can assume that $r = d(a, C) > 0$. By definition of infimum, there exists $x_n \in C$ such that $d(a, x_n) \leq (1 + \frac{1}{n})r$ for every $n \geq 1$. We are going to prove that $(x_n)_n$ is a Cauchy sequence. Assume otherwise that the sequence $(x_n)_n$ is not Cauchy. Then there exist $\varepsilon_0 > 0$ and two subsequences $(x_{n_k})_k$ and $(x_{m_k})_k$ of $(x_n)_n$ such that $n_k > m_k$, $d(x_{n_k}, x_{m_k}) \geq \varepsilon_0$ for any $k \geq 1$. We have

$$d(a, x_{m_k}) \leq (1 + 1/m_k)r, \quad d(a, x_{n_k}) \leq (1 + 1/n_k)r < (1 + 1/m_k)r,$$

and

$$d(x_{m_k}, x_{n_k}) \geq \varepsilon_0 \geq \Big(1 + \frac{1}{m_k}\Big) r \frac{\varepsilon_0}{2r}$$

for any $k \geq 1$. Since X is UC, there is $\eta(r, \frac{\varepsilon_0}{2r}) < \delta(a, (1 + 1/m_k)r, \frac{\varepsilon_0}{2r})$ such that

$$d\Big(a, \frac{1}{2}x_{n_k} \oplus \frac{1}{2}x_{m_k}\Big) < \Big(1 + \frac{1}{m_k}\Big) r \Big(1 - \eta(r, \frac{\varepsilon_0}{2r})\Big).$$

for every $k \geq 1$. We note that $\frac{1}{2}x_{n_k} \oplus \frac{1}{2}x_{m_k} \in C$ since C is convex. Thus for any $k \geq 1$,

$$r < \Big(1 + \frac{1}{m_k}\Big) r(1 - \eta(r, \frac{\varepsilon_0}{2r})).$$

Letting $k \to \infty$, we obtain a contradiction since $r \leq r(1 - \eta(r, \frac{\varepsilon_0}{2r}))$ with $r > 0$ and $\eta(r, \frac{\varepsilon_0}{2r}) \in (0, 1)$. Hence $(x_n)_n$ is a Cauchy sequence. Thus there exists $c \in X$ such that $\lim_{n\to\infty} d(c, x_n) = 0$. It implies that $c \in C$ since C is closed. For each $n \geq 1$, we have

$$r = d(a, C) \leq d(a, c) \leq d(a, x_n) + d(c, x_n) \leq (1 + \frac{1}{n})r + d(c, x_n).$$

Letting $n \to \infty$, we conclude that $d(a, C) = d(a, c)$.

Next we are going to prove the uniqueness of c. Assume that there exists $c' \in C$ such that $c' \neq c$ and $d(a, c') = r$. Put $r_1 = d(c, c')$ and $\varepsilon = \frac{r_1}{r}$. Since X is UC, we have

$$d\Big(a, \frac{1}{2}c \oplus \frac{1}{2}c'\Big) \leq r(1 - \delta(a, r, \varepsilon)).$$

Since $\frac{1}{2}x_0 \oplus \frac{1}{2}x_1 \in C$, we have $r \leq r(1 - \delta(a, r, \varepsilon))$. This is a contradiction with $r > 0$ and $\delta(a, r, \varepsilon) > 0$. Therefore, c is the unique point such that $d(a, c) = d(a, C)$. □

Similarly, the point (i) of the following lemma has been proved in [70] in the case of hyperbolic spaces, and the point (ii) is a counterpart of Proposition 3.5 in [1] for modular spaces.

Lemma 4.1.6. *Let (X, d) be a complete uniquely geodesic metric space. Assume that X is UC. Then the following properties hold:*

(i) Any nonincreasing sequence $(C_n)_{n\geq 1}$ of nonempty bounded closed convex subsets of X has a nonempty intersection.

(ii) Any family of nonempty closed bounded convex subsets of X satisfying the finite intersection property has nonempty intersection.

Proof. (i) Suppose that $(C_n)_{n\geq 1}$ is a nonincreasing sequence of nonempty bounded closed convex subsets of X. If $C_n = X$ for all $n \geq 1$, then we are done. So we assume that $C_{n_0} \neq X$ for some $n_0 > 1$ and take $x \in X \setminus C_{n_0}$. It is not difficult to see that the sequence $(d(x, C_n))_n$ is nondecreasing and bounded. Hence there exists the limit $r = \lim\limits_{n\to\infty} d(x, C_n)$. Clearly, $r \in (0, \infty)$. It follows from Theorem 4.1.5 that for each $n \geq 1$, there exists $x_n \in C_n$ such that $d(x, C_n) = d(x, x_n)$. Since $(C_n)_n$ is non-increasing, we have that $x_k \in C_n$ for any $k \geq n$. Using a similar argument as in the proof of Theorem 4.1.5, there exists $x_0 \in X$ such that $\lim\limits_{n\to\infty} d(x_n, x_0) = 0$. Since C_n is closed, $x_0 \in C_n$ for all $n \geq 1$, i.e., $x_0 \in \bigcap\limits_{n\geq 1} C_n$.

(ii) Suppose that $(Y_i)_{i\in I}$ is a family of nonempty bounded closed convex subsets of X such that $\bigcap_{i\in F} Y_i \neq \emptyset$ for any finite subset F of I. We fix $i_0 \in I$, and put $C_i := Y_i \cap Y_{i_0}$ for each $i \in I$. We only need to prove that $\bigcap_{i\in I} C_i \neq \emptyset$. Obviously, $(C_i)_{i\in I}$ is a family of nonempty bounded closed convex subsets of Y_{i_0} satisfying the finite intersection property.

Put

$$\mathcal{J} = \{J \subseteq I : J \text{ is countable}\}.$$

First we are going to prove that if $J \in \mathcal{J}$, then $\bigcap\limits_{j\in J} C_j \neq \emptyset$. Indeed, assume that $J = \{j_1, j_2, ...\}$. For each $n \geq 1$, put $J(n) = \{j_1, ..., j_n\}$. Let $A_n = \bigcap\limits_{j\in J(n)} C_j$ for any $n \geq 1$. It is not difficult to see that $(A_n)_n$ is a decreasing sequence of nonempty bounded closed convex subsets of X. Using (i), we have $C_J = \bigcap\limits_{j\in J} C_j \neq \emptyset$.

Take $x \in X$. For each $J \in \mathcal{J}$, we put $d_J := d(x, C_J)$ and

$$d_{\mathcal{J}} = \sup\{d_J : J \in \mathcal{J}\}.$$

Clearly, $d_{\mathcal{J}} \in [0, \infty)$. For any $n \geq 1$, there exists a subset $J_n \in \mathcal{J}$ such that

$$d_{\mathcal{J}} - \frac{1}{n} \leq d_{J_n} \leq d_{\mathcal{J}}.$$

For each $n \geq 1$, put $J_n^* = \bigcup\limits_{i=1}^{n} J_i$. Clearly, J_n^* is countable. Thus $\Big(\bigcap\limits_{j\in J_n^*} C_j\Big)_n$ is a decreasing sequence of nonempty bounded closed convex subsets of C_{i_0}. It follows from (i) that $K = \bigcap\limits_{j\in F} C_j \neq \emptyset$, where $F = \bigcup\limits_{n\geq 1} J_n^* = \bigcup\limits_{n\geq 1} J_n$. We note that $\bigcup\limits_{n\geq 1} J_n$ is a countable subset of I, i.e, $\bigcup\limits_{n\geq 1} J_n \in \mathcal{J}$. Hence

$$d_{\mathcal{J}} - \frac{1}{n} \leq d_{J_n} \leq d(x, K) \leq d_{\mathcal{J}}$$

for any $n \geq 1$. It implies $d(x, K) = d_{\mathcal{J}}$. Now Theorem 4.1.5 yields the existence of a unique $y \in K$ such that $d(x, y) = d(x, K) = d_{\mathcal{J}}$.

Take $i \in I$. Since $F \cup \{i\}$ is countable, $K \cap C_i \neq \emptyset$ and

$$d(x, K) \leq d(x, K \cap C_i) \leq d_{\mathcal{J}}.$$

Hence $d(x, K) = d(x, K \cap C_i) = d_{\mathcal{J}}$, which implies $y \in K \cap C_i$. Thus $y \in C_i$ for every $i \in I$, that is, $y \in \bigcap_{i \in I} C_i$. □

Remark 4.1.7. We can prove in Lemma 4.1.6 that (i) is equivalent to (ii). Indeed, we only have to prove that (ii) implies (i). Suppose that (ii) holds and take a decreasing sequence $(C_n)_n$ of nonempty bounded closed bounded convex subsets of X. Then for any finite subset $\{n_1, ..., n_l\} \subset \mathbb{N}$, where $n_1 \leq ... \leq n_l$, we have $\bigcap_{k=1}^{l} C_{n_k} = C_{n_l} \neq \emptyset$. Hence $(C_n)_n$ has the finite intersection property and thus $\bigcap_{n \in \mathbb{N}} C_n \neq \emptyset$.

Lemma 4.1.6 allows us to use normal structures in UC metric spaces. Let us recall definitions of normal structure and uniform normal structure (see [78]).

Definition 4.1.8. A convex structure in a metric space X is a family $\mathcal{F}$ of subsets of X such that $\emptyset, X, \{x\} \in \mathcal{F}$ for every $x \in X$, and $\mathcal{F}$ is closed under arbitrary intersections. The structure $\mathcal{F}$ is said to be compact if every subfamily of $\mathcal{F}$ which has the finite intersection property has nonempty intersection.

Given a convexity structure $\mathcal{F}$ in a metric space (X, d), we adopt the following notation: for $D \in \mathcal{F}$ and $x \in X$, set

$$\begin{aligned} r_x(D) &= \sup\{d(x, y) : y \in D\}, \\ r_X(D) &= \inf\{r_x(D) : x \in X\}, \\ r(D) &= \inf\{r_x(D) : x \in D\}. \end{aligned}$$

Definition 4.1.9. We say that X has normal structure (resp. uniform normal structure) if there exists a convexity structure $\mathcal{F}$ on X such that $r(A) < \operatorname{diam}(A)$ (resp. $r(A) \leq c \operatorname{diam}(A)$ for a fixed constant $c \in (0, 1)$) for any nonempty $A \in \mathcal{F}$ which is bounded and not reduced to a single point. We will also say that $\mathcal{F}$ is normal (resp. uniformly normal).

A subset A of a metric space X is said to be admissible if A is the intersection of closed balls centered at points of X. Of particular interest in metric fixed point theory is the convexity structure $\mathcal{A}(X)$ consisting of $\emptyset$, X and all admissible sets in X. Given any bounded set $A \subseteq X$, set

$$cov(A) := \bigcap \{D : D \in \mathcal{A}(X) \text{ and } D \supseteq A\}.$$

Clearly, $cov(A) \in \mathcal{A}(X)$ and thus $A = cov(A) \Leftrightarrow A \in \mathcal{A}(X)$.

Lemma 4.1.10 ([101]). *Let X be a complete UC metric space. Then X has normal structure.*

Proof. Let $\mathcal{F}$ be the family consisting of $\emptyset$, X and all nonempty closed convex bounded subsets of X. Since X is UC, $\mathcal{F}$ is a compact convexity structure. We are going to prove that $r(A) < \operatorname{diam}(A)$ for any $A \in \mathcal{F}$ which is not reduced to a single point. Assume that $A \in \mathcal{F}$ and A has at least two distinct elements. Denote $d = \operatorname{diam}(A)$, $r = r(A)$. By definition of the diameter of A, we can choose $x, y \in A$ such that $d(x, y) \geq d/2$. Let $w = \frac{1}{2}x \oplus \frac{1}{2}y$. For every $z \in A$, we have $d(z, x) \leq d$, $d(z, y) \leq d$ and $d(x, y) \geq d/2$. Since X is UC, it follows that

$$d(z, w) \leq d - d\delta(z, d, 1/2) \leq d - d\eta(d/2, 1/2),$$

and so

$$r_w(A) \leq d - d\eta(d/2, 1/2).$$

Thus

$$r \leq d - d\eta(d/2, 1/2),$$

and since $\eta(d/2, 1/2) > 0$, we have $r < d$, i.e., $r(A) < \operatorname{diam}(A)$. Therefore, X has normal structure. □

4.2. Fixed points of monotone G-nonexpansive mappings

In this section, we present some fixed point theorems for monotone G-nonexpansive mappings. The setting are reflexive metric spaces, in particular, uniformly convex metric spaces. It is well-known that a Banach space is said to be reflexive if every nonincreasing family of nonempty bounded closed convex subsets has nonempty intersection. Thus it makes sense to define reflexivity for metric spaces as follows:

Definition 4.2.1 ([63]). Let I be a directed set. A complete geodesic metric space X is said to be reflexive if for every nonincreasing family $(C_i)_{i\in I}$ of nonempty, bounded, closed, convex subsets, i.e., $C_i \subset C_j$ whenever $j \leq i$, then

$$\bigcap_{i\in I} C_i \neq \emptyset.$$

Lemma 4.2.2 ([63]). *A space X is reflexive if and only if any family of nonempty, closed, bounded, convex subsets of X satisfying the finite intersection property has nonempty intersection.*

Our first result is an application of Theorem 3.2.5 to the case of a reflexive metric space with a partial order $\preceq := E(G)$.

Theorem 4.2.3 ([101]). *Let $(X, d, \preceq)$ be a reflexive metric space with a partial order $\preceq$, and C be a nonempty bounded closed convex subset of X. Assume that order intervals are closed and convex. Let $T : C \to C$ be a monotone mapping. If there exists $c \in C$ such that $c \preceq T(c)$, then T has a fixed point.*

Proof. Let $\mathcal{G}$ be the collection of all subsets of the form $C \cap P$, where P is an order interval in X. By Lemma 4.2.2, $\mathcal{G}$ satisfies that any subcollection $\mathcal{G}'$ of $\mathcal{G}$ having the finite intersection property, has nonempty intersection. It follows from Theorem 3.2.5 that there exists $s \in C$ such that $T([s, s]_G) \subseteq [s, s]_G$. Since $\preceq$ is a partial order, $[s, s]_G$ is a singleton and hence T has a fixed point in C. □

Since nearly uniformly convex metric spaces (in the sense of Kell, see [63, Definition 2.2]) are reflexive, we have the following corollary.

Corollary 4.2.4. *Let $(X, d, \preceq)$ be a nearly uniformly convex metric space with a partial order $\preceq$, and let C be a bounded closed convex subset of X. Assume that order intervals are closed and convex. Let $T : C \to C$ be a monotone mapping. If there exists $c \in C$ such that $c \preceq T(c)$, then T has a fixed point.*

In 2016, Dehaish and Khamsi [38] showed that if C is a bounded closed and convex subset of a partially ordered uniformly convex hyperbolic metric space, then every monotone nonexpansive mapping $T : C \to C$ has a fixed point. By Lemma 4.1.6, we show an analogue of Corollary 4.2.4 for UC metric spaces, thus giving a wide generalization of Dehaish-Khamsi's theorem by dropping both assumptions about hyperbolicity of the space and nonexpansivity of the mapping.

Theorem 4.2.5 ([101]). *Let X be a complete uniquely geodesic metric space with a partial order $\preceq$. Assume that X is UC, and order intervals are convex and closed. Let C be a nonempty bounded closed convex subset of X and let $T : C \to C$ be a monotone mapping. If there exists $c \in C$ such that $c \preceq T(c)$, then T has a fixed point in C.*

Proof. It is enough to notice that from Lemma 4.1.6 (ii) X is reflexive and then apply Theorem 4.2.3. □

The next results extend Theorem 4.2.3 for monotone G-monotone mappings in reflexive metric spaces with digraphs.

Definition 4.2.6. Let (X, d) be a metric space endowed with a digraph G such that $V(G) \subseteq X$. A map $T : X \to X$ is said to be monotone G-monotone mapping if T is G-monotone and satisfies

$$d(T(x), T(y)) \le d(x, y),$$

for any $x, y \in X$ such that $y \in [x, \to)_G$.

Theorem 4.2.7 ([101]). *Let X be a reflexive metric space with a digraph G and let C be a bounded closed convex subset of X. Assume that G-intervals along walks are closed and convex, and for each $a \in C$, $[a, a]_G$ is either empty or has the fixed point property for nonexpansive mappings. If $T : C \to C$ is monotone G-nonexpansive and there exists $c \in C$ such that $T(c) \in [c, \to)_G$, then T has a fixed point in C.*

Proof. It follows from Theorem 3.2.5 that there exists $s \in C$ such that $[s, s]_G \neq \emptyset$, $T([s, s]_G) \subseteq [s, s]_G$ and T is nonexpansive on $[s, s]_G$ since $x \in [y, \to)_G$ and $y \in [x, \to)_G$ for any $x, y \in [s, s]_G$. By assumption, T has a fixed point in $[s, s]_G$. □

Corollary 4.2.8 ([101]). *Let X be a complete UC metric space with a digraph G. Assume that G-intervals along walks are convex and closed. Let C be a nonempty bounded closed and convex subset of X. If $T : C \to C$ is monotone G-nonexpansive and there exists $c \in C$ such that $T(c) \in [c, \to)_G$, then T has a fixed point in C.*

Proof. It follows from Lemma 4.1.6 (ii) that X is reflexive. Without loss of generality we can assume that $V(G) = C$. It is sufficient to prove that each nonempty $[a,a]_G, a \in C$, has the fixed point property for nonexpansive mappings. Fix such $[a,a]_G$, and let $\mathcal{F}$ be the family consisting of $\emptyset$ and all bounded closed convex subsets of $[a,a]_G$. By virtue of Lemma 4.1.6 (ii), $\mathcal{F}$ is a convexity structure on $[a,a]_G$, and $\mathcal{F}$ is also compact. We invoke Lemma 4.1.10 to deduce that $[a,a]_G$ has normal structure. Applying Theorem 3.2 in [78] (see also [64, Theorem 8]) we conclude that T has a fixed point in $[a,a]_G$. Now the conclusion follows from Theorem 4.2.7. □

Remark 4.2.9. Notice that the set $\mathrm{Fix}(T)_{[a,a]_G}$ of fixed points of T in $[a,a]_G$ is closed and convex if X is a complete UC metric space. Indeed, to show that $\mathrm{Fix}(T)_{[a,a]_G}$ is closed, select a sequence $(x_n)_n$ in $\mathrm{Fix}(T)_{[a,a]_G}$ which converges to $x \in [a,a]_G$. Then

$$d(x_n, T(x)) = d(Tx_n, T(x)) \leq d(x_n, x) \text{ for all } n,$$

and hence $(x_n)_{n\geq 1}$ also converges to $T(x)$. By the uniqueness of the limit, $x = T(x)$. Thus $x \in \mathrm{Fix}(T)_{[a,a]_G}$ and therefore, $\mathrm{Fix}(T)_{[a,a]_G}$ is closed.

To show convexity, let $x, y \in \mathrm{Fix}(T)_{[a,a]_G}$ with $x \neq y$ and set $2r = d(x,y) > 0$. We prove that $z = \frac{1}{2}x \oplus \frac{1}{2}y \in \mathrm{Fix}(T)_{[a,a]_G}$. Assume conversely that $z \neq T(z)$ and let $d(z, T(z)) = r_0$. We have $d(x,z) = \frac{1}{2}d(x,y) = r$ and

$$d(x, T(z)) = d(T(x), T(z)) \leq d(x,z),\ d(z, T(z)) = r\frac{r_0}{r}.$$

Hence

$$d(x, \frac{1}{2}z \oplus \frac{1}{2}T(z)) \leq r\Big(1 - \delta(r, \frac{r_0}{r})\Big),$$

and similarly,

$$d(y, \frac{1}{2}z \oplus \frac{1}{2}T(z)) \leq r\Big(1 - \delta(r, \frac{r_0}{r})\Big).$$

By the triangle inequality,

$$2r = d(x,y) \leq 2r - r\Big(\delta(r, \frac{r_0}{r}) + \delta(r, \frac{r_0}{r})\Big) < 2r,$$

and we obtain a contradiction. Therefore, $z = T(z)$. This shows that $\mathrm{Fix}(T)_{[a,a]_G}$ is convex.

We are thus led to the following theorem.

Theorem 4.2.10 ([101]). *Let X be a complete UC metric space with a digraph G. Assume that G-intervals along walks are convex and closed. Let C be a bounded closed and convex subset of X. Let $T_1, T_2 : C \to C$ be two monotone G-nonexpansive mappings which are commutative. If there exists $c \in C$ such that $T_i(c) \in [c, \rightarrow)_G$ for $i = 1,2$, then* $\mathrm{Fix}(T_1) \cap \mathrm{Fix}(T_2)$ *is nonempty.*

Proof. Without loss of generality we can assume that $V(G) = C$. Arguing in a similar way to the proof of Theorem 3.2.5 there exists $s \in C$ such that $T_i([s,s]_G) \subseteq [s,s]_G$, and T_i are nonexpansive on $[s,s]_G$ for $i = 1,2$. By Corollary 4.2.8 and Remark 4.2.9, $\mathrm{Fix}(T_1)_{[s,s]_G}$ and $\mathrm{Fix}(T_2)_{[s,s]_G}$ are nonempty, closed and convex. Since T_1, T_2 are commutative, we have $T_2(\mathrm{Fix}(T_1)_{[s,s]_G}) \subseteq \mathrm{Fix}(T_1)_{[s,s]_G}$. Hence $T_2 : \mathrm{Fix}(T_1)_{[s,s]_G} \to \mathrm{Fix}(T_1)_{[s,s]_G}$ has a fixed point in $\mathrm{Fix}(T_1)_{[s,s]_G}$ by Corollary 4.2.8. It follows that $\mathrm{Fix}(T_1)_{[s,s]_G} \cap \mathrm{Fix}(T_2)_{[s,s]_G}$ is nonempty, bounded, closed and convex. Hence $\mathrm{Fix}(T_1) \cap \mathrm{Fix}(T_2)$ is nonempty. □

Remark 4.2.11. Note that the conclusion of Theorem 4.2.10 holds for a finite family of monotone G-nonexpansive mappings which are commutative.

By Remark 4.2.11 and using Lemma 4.1.6 (ii), we can extend Theorem 4.2.10 for any commutative family of monotone G-nonexpansive mappings.

Theorem 4.2.12 ([101]). *Let X be a complete UC metric space with a digraph G. Assume that G-intervals along walks are convex and closed. Let C be a nonempty bounded closed convex of X. Let $\mathcal{T}$ be a commutative family of monotone G-nonexpansive mappings from C into C. If there exists $c \in C$ such that $T(c) \in [c, \to)_G$ for every $T \in \mathcal{T}$, then $\bigcap_{T\in\mathcal{T}} \mathrm{Fix}(T)$ is nonempty.*

Proof. By Theorem 3.5.1, there exists $s \in C$ such that $T([s, s]_G) \subseteq [s, s]_G$ for any $T \in \mathcal{T}$. By Corollary 4.2.8 and Remark 4.2.9, $\mathrm{Fix}(T)_{[s,s]_G}$ are nonempty, closed and convex. Because of the above argument the family $(\mathrm{Fix}(T)_{[s,s]_G})_{T\in\mathcal{T}}$ satisfies the finite intersection property. It implies that $\bigcap_{T\in\mathcal{T}} \mathrm{Fix}(T)_{[s,s]_G} \neq \emptyset$. ☐

4.3. Fixed points of monotone G-nonexpansive multivalued mappings

Let (X, d) be a metric space. A multivalued mapping $T : X \to 2^X$ is said to be nonexpansive if for each $x, y \in X$, $d_H(T(x), T(y)) \leq d(x, y)$, where d_H is the Hausdorff metric.

In 1968, Markin [89] proved that in a Hilbert space, any nonexpansive multivalued mapping $T : C \to \mathbb{CP}(C)$ possesses a fixed point under the condition that C is a nonempty weakly compact convex subset, $\mathbb{CP}(C)$ is the family of compact subsets of C. Later, Browder [29] proved a similar result for spaces with weakly continuous duality mapping, and Lami Dozo [46] proved it for spaces satisfying Opial's condition. Then Assad and Kirk [16] generalized Lami Dozo's result. In 1974, Lim [85] showed a fixed point theorem by considering a bounded closed and convex subset of a uniformly convex Banach space. It is natural to give a counterpart of Lim's theorem for monotone G-nonexpansive multivalued mappings in UC hyperbolic metric spaces.

Definition 4.3.1 ([9]). Let (X, d) be a metric space with a digraph G, C be a nonempty subset of X, and 2^C be a family of nonempty subsets of C. A multivalued mapping $T : C \to 2^C$ is said to be monotone G-nonexpansive if for any $x, y \in C$ with $(x, y) \in E(G)$ and any $u \in T(x)$, there exists $v \in T(y)$ such that

$$(u, v) \in E(G), \quad \text{and} \quad d(u, v) \leq d(x, y).$$

Assume that $(x_n)_n$ is a bounded sequence in a hyperbolic metric space (X, d), C is a nonempty subset of X. The type function $\tau : C \to [0, \infty)$ is defined as follows:

$$\tau(x) = \limsup_{n\to\infty} d(x_n, x) \tag{4.3.1}$$

for all $x \in C$.

The following properties of type function are necessary for our results.

Lemma 4.3.2 ([37]). *Let (X, d) be a uniformly convex hyperbolic metric space, $(x_n)_n \subset X$ a bounded sequence, and C a closed convex subset of X. Assume that τ is the type function defined by $(x_n)_n$ as in (4.3.1). Then*

(i) The function τ is continuous and convex;

(ii) There exists a unique minimum point $c \in C$ such that

$$\tau(c) = \inf_{x \in C} \tau(x).$$

Define

$$r(C, (x_n)) = \inf \left\{ \limsup_{n \to \infty} d(x_n, x) : x \in C \right\}.$$

In what follows, we show an analogue of Lemma 15.2 [54] for hyperbolic metric spaces.

Lemma 4.3.3. *Let X be a metric space, C a nonempty subset of X and $(x_n)_n$ a bounded sequence in X. Then there exists a subsequence $(x_{n_k})_k$ of $(x_n)_n$ such that for every subsequence $(x_{n_{k_l}})_l$ of $(x_{n_k})_k$,*

$$r(C, (x_{n_k})) = r(C, (x_{n_{k_l}})).$$

Proof. If $(y_n)_n$ is a subsequence of $(x_n)_n$, we will use the notation $(y_n) \prec (x_n)$. Define

$$r_0 := \inf \left\{ r(C, (y_n)) : (y_n) \prec (x_n) \right\}.$$

Then we can choose $(y_n^1) \prec (x_n)$ such that

$$r(C, (y_n^1)) < r_0 + 1.$$

Define

$$r_1 := \inf \left\{ r(C, (z_n)) : (z_n) \prec (y_n^1) \right\},$$

and select $(y_n^2) \prec (y_n^1)$ such that

$$r(C, (y_n^2)) < r_1 + \frac{1}{2}.$$

Continuing this process, we can construct sequences (y_n^i) with

$$r_i := \inf \left\{ r(C, (z_n)) : (z_n) \prec (y_n^i) \right\}$$

such that $(y_n^i) \prec (y_n^{i-1})$ and

$$r(C, (y_n^{i+1})) < r_i + \frac{1}{i+1}$$

for any $i \geq 1$. Since $(r_i)_i$ is nondecreasing and bounded from above by $r(C, (x_n))$, it has a limit, say r. Hence $\lim_{i \to \infty} r(C, (y_n^{i+1})) = r$.

Now take the diagonal sequence (y_n^n) and denote $\overline{r} = r(C, (y_n^n))$. Then $(y_n^n) \prec (y_n^i)$, and hence $\overline{r} \geq r_i$. On the other hand, we have $(y_n^n) \prec (y_n^{i+1})$, which gives $\overline{r} \leq r_i + \frac{1}{i+1}$. Thus $\overline{r} = r$.

Since any subsequence (u_n) of (y_n^n) also satisfies (for the same reasons) the inequalities

$$r(C,(u_n) \geq r_i \quad \text{and} \quad r(C,(u_n)) \leq r_i + \frac{1}{i+1}$$

for any $i \geq 1$, one gets $r(C,(u_n)) = r$. We conclude that $(y_n^n)_n$ is the desired subsequence. □

We also need the following important proposition. It is a consequence of the result of Goebel and Kirk [53, Proposition 1].

Proposition 4.3.4. *Let (X,d) be a hyperbolic metric space. Let $(x_n)_n$ and $(y_n)_n$ be two sequences in (X,d) such that*

$$x_{n+1} = \frac{1}{2}x_n \oplus \frac{1}{2}y_n$$

for any $n \geq 1$. Suppose that

$$d(y_n, y_{n+1}) \leq d(x_n, x_{n+1}), \quad n \geq 1.$$

Then we have

$$(1+\frac{n}{2})d(x_i,y_i) \leq d(x_i,y_{i+n}) + 2^n\Big(d(x_i,y_i) - d(x_{i+n},y_{i+n})\Big) \tag{4.3.2}$$

for every $i,n \geq 1$. In particular, if $(x_n)_n$ and $(y_n)_n$ are bounded then $\lim_{n\to\infty} d(x_n,y_n) = 0$.

Theorem 4.3.5. *Let (X,d) be a complete hyperbolic metric space with a transitive digraph G. Assume that X is UC and G-intervals are closed and convex. Let C be a nonempty bounded closed convex subset of X. Let $T : C \to \mathbb{CP}(C)$ be a monotone G-nonexpansive multivalued mapping. If there exists $x_0 \in C$ such that $(x_0, y_0) \in E(G)$ for some $y_0 \in T(x_0)$, then* $\mathrm{Fix}(T) \neq \emptyset$.

Proof. Put

$$x_1 = \frac{1}{2}x_0 \oplus \frac{1}{2}y_0.$$

Since G-intervals are convex and $(x_0,y_0) \in E(G)$, we have $(x_0,x_1),(x_1,y_0) \in E(G)$. Since T is a monotone G-nonexpansive multivalued mapping, there is $y_1 \in T(x_1)$ such that

$$(y_0,y_1) \in E(G) \quad \text{and} \quad d(y_1,y_0) \leq d(x_1,x_0).$$

Continuing in this manner, we can construct sequences $(x_n)_n$ and $(y_n)_n$ in C, defined as follows:

$$x_{n+1} = \frac{1}{2}x_n \oplus \frac{1}{2}y_n, \quad y_n \in T(x_n) \quad \text{for all } n \geq 0. \tag{4.3.3}$$

By induction, we have

$$(x_n,x_{n+1}),(x_{n+1},y_n),(y_n,y_{n+1}) \in E(G)$$

and

$$d(y_n,y_{n+1}) \leq d(x_n,x_{n+1}) = \frac{1}{2}d(x_n,y_n)$$

for any $n \geq 0$. We have

$$\begin{aligned} d(x_{n+1}, y_{n+1}) &\leq d(x_{n+1}, y_n) + d(y_n, y_{n+1}) \\ &\leq \frac{1}{2}d(x_n, y_n) + d(x_n, x_{n+1}) = \frac{1}{2}d(x_n, y_n) + \frac{1}{2}d(x_n, y_n) = d(x_n, y_n). \end{aligned}$$

Hence $d(x_{n+1}, y_{n+1}) \leq d(x_n, y_n)$ for every $n \geq 0$. By Proposition 4.3.4 we have that $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Since G is transitive and $(x_n)_n$ is nondecreasing, $([x_n, \to)_G, n \geq 0)_n$ is a nonincreasing sequence of nonempty closed convex subsets of X. It follows from Lemma 4.1.6 that

$$C_\infty = \bigcap_{n\geq 0} [x_n, \to)_G \cap C \neq \emptyset.$$

Now Lemma 4.3.3 implies the existence of a subsequence $(x_{n_k})_k$ of $(x_n)_n$ such that for each subsequence $(x_{n_{k_l}})_l$ of $(x_{n_k})_k$ we have

$$r(C_\infty, (x_{n_{k_l}})) = r(C_\infty, (x_{n_k})).$$

From Lemma 4.3.2 there exists a unique $c \in C_\infty$ such that

$$\limsup_{k\to\infty} d(x_{n_k}, c) = \inf\{\limsup_{k\to\infty} d(x_{n_k}, x) : x \in C_\infty\} = r(C_\infty, (x_{n_k})).$$

Thus we have $(x_{n_k}, c) \in E(G)$ for any $k \geq 1$. Since T is a monotone G-nonexpansive multivalued mapping, there exists $c_{n_k} \in T(c)$ such that

$$(y_{n_k}, c_{n_k}) \in E(G) \quad \text{and} \quad d(y_{n_k}, c_{n_k}) \leq d(x_{n_k}, c)$$

for any $k \geq 1$. Since $T(c)$ is compact, there exists a subsequence $(c_{n_{k_l}})_l$ of $(c_{n_k})_k$ such that $\lim_{l\to\infty} c_{n_{k_l}} = c' \in T(c)$. First we prove that $c' \in C_\infty$. Indeed, it is not difficult to see that

$$\emptyset \neq \bigcap_{k\geq 0} [y_{n_k}, \to)_G \subseteq \bigcap_{k\geq 0} [x_{n_k}, \to)_G = \bigcap_{n\geq 0} [x_n, \to)_G = C_\infty.$$

For each $m \geq 0$ and $n_{k_l} \geq m$, we have

$$(y_m, y_{m+1}), (y_{n_{k_l}}, c_{n_{k_l}}) \in E(G).$$

It implies that $c_{n_{k_l}} \in [y_m, \to)_G$ for any $n_{k_l} \geq m$. Hence $c' \in [y_m, \to)_G$ for every $m \geq 0$. Thus $c' \in \bigcap_{m\geq 0} [y_m, \to)_G$ and therefore,

$$c' \in \bigcap_{m\geq 0} [x_m, \to)_G.$$

Now, we are going to prove that $c = c'$. We have

$$\begin{aligned} d(x_{n_{k_l}}, \frac{1}{2}c \oplus \frac{1}{2}c') &\leq \frac{1}{2}d(x_{n_{k_l}}, c) + \frac{1}{2}d(x_{n_{k_l}}, c') \\ &\leq \frac{1}{2}d(x_{n_{k_l}}, c) + \frac{1}{2}\Big(d(x_{n_{k_l}}, y_{n_{k_l}}) + d(y_{n_{k_l}}, c_{n_{k_l}}) + d(c_{n_{k_l}}, c')\Big). \end{aligned}$$

Taking upper limit as $l \to \infty$, we get

$$\begin{aligned}\limsup_{l\to\infty} d(x_{n_{k_l}}, \frac{1}{2}c \oplus \frac{1}{2}c') &\le \frac{1}{2}\limsup_{l\to\infty} d(x_{n_{k_l}}, c) + \frac{1}{2}\limsup_{l\to\infty} d(y_{n_{k_l}}, c_{n_{k_l}}) \\ &\le \frac{1}{2}\limsup_{l\to\infty} d(x_{n_{k_l}}, c) + \frac{1}{2}\limsup_{l\to\infty} d(x_{n_{k_l}}, c) \\ &= \limsup_{l\to\infty} d(x_{n_{k_l}}, c) \le \limsup_{k\to\infty} d(x_{n_k}, c) = r(C_\infty, (x_{n_k}))\end{aligned}$$

since $d(y_{n_{k_l}}, c_{n_{k_l}}) \le d(x_{n_{k_l}}, c)$ for any positive integer l. From convexity of C_∞ we have

$$\limsup_{l\to\infty} d(x_{n_{k_l}}, \frac{1}{2}c \oplus \frac{1}{2}c') \ge r(C_\infty, (x_{n_{k_l}})).$$

Thus

$$r(C_\infty, (x_{n_k})) = r(C_\infty, (x_{n_{k_l}})) \le \limsup_{l\to\infty} d(x_{n_{k_l}}, \frac{1}{2}c \oplus \frac{1}{2}c') \le r(C_\infty, (x_{n_k})).$$

Hence

$$r(C_\infty, (x_{n_{k_l}})) = \limsup_{l\to\infty} d(x_{n_{k_l}}, \frac{1}{2}c \oplus \frac{1}{2}c') = r(C_\infty, (x_{n_k})).$$

On the other hand, we have

$$r(C_\infty, (x_{n_{k_l}})) \le \limsup_{l\to\infty} d(x_{n_{k_l}}, c) \le \limsup_{k\to\infty} d(x_{n_k}, c) = r(C_\infty, (x_{n_k})).$$

Thus

$$r(C_\infty, (x_{n_{k_l}})) = \limsup_{l\to\infty} d(x_{n_{k_l}}, c).$$

By uniqueness, we have $c = \frac{1}{2}c \oplus \frac{1}{2}c'$, i.e., $c = c'$. Therefore, c is a fixed point of T. □

Example 4.3.6. Consider the closed interval $[0,1] \subset \mathbb{R}$ with the absolute value $|\cdot|$. Let $G = (V(G), E(G))$ such that

$$V(G) = [0, \frac{1}{2}] \quad \text{and} \quad E(G) = \{(x,y) : x, y \in [0, \frac{1}{2}]\}.$$

Then G is reflexive, transitive, and G-intervals are closed, convex. Define a multivalued mapping $T : [0,1] \to \mathbb{CP}([0,1])$ by

$$T(x) = \begin{cases} \{x/2, x/3\} & \text{if } x \neq 1 \\ \{0\} & \text{if } x = 1. \end{cases}$$

We have

$$d_H(T(1), T(3/4)) = 3/8 > | 1 - 3/4 | .$$

Thus T is not nonexpansive. However, it is not difficult to see that T is monotone G-nonexpansive and $\mathrm{Fix}(T) = \{0\}$.

By putting $E(G) := \preceq$, we obtain the following result.

Corollary 4.3.7. *Let* $(X, d, \preceq)$ *be a complete hyperbolic metric space with a partial order* $\preceq$*. Assume that* (X, d) *is UC, and all order intervals are closed and convex. Let* C *be a nonempty bounded closed convex subset of* X*. Let* $T : C \to \mathbb{CP}(C)$ *be a monotone nonexpansive multivalued mapping. Assume that there exists* $x_0 \in C$ *such that* $x_0 \preceq y_0$ *for some* $y_0 \in T(x_0)$*. Then there exists* $c \in X$ *such that* $c \in T(c)$*.*

In the case of single-valued mappings, as a consequence of Theorem 4.3.5, we obtain Theorem 3.3 in [15] of Alfuraidan and Shukri.

Corollary 4.3.8. *Let* (X, d) *be a complete hyperbolic metric space with a reflexive, transitive digraph* G*. Assume that* X *is UC and G-intervals are closed and convex. Let* C *be a nonempty bounded closed convex subset of* X*. Let* $T : C \to C$ *be a monotone G-nonexpansive mapping. Then* T *has a fixed point provided that there exists* $x_0 \in C$ *such that* $(x_0, T(x_0)) \in E(G)$*.*

Obviously, if X is p-UC hyperbolic metric space in the sense of Quan, then X is UC. Therefore, Theorem 4.3.5 is an extension of Quan's theorem.

Lemma 4.3.9 ([99]). *Let* (X, d) *be a complete hyperbolic metric space with a reflexive, transitive digraph* G*. Assume that* X *is p-UC and G-intervals are closed and convex. Let* C *be a nonempty, closed, convex and bounded subset of* X*. Let* $T : C \to \mathbb{CP}(C)$ *be a monotone G-nonexpansive multivalued mapping. Then* $\mathrm{Fix}(T) \neq \emptyset$ *provided there exists* $x_0 \in C$ *such that* $(x_0, y_0) \in E(G)$ *for some* $y_0 \in T(x_0)$*.*

Chapter 5

Fixed points of G-monotone mappings in modular spaces

We know that modular functions do not adhere to metric properties. However, modular spaces are constructed based on linear vector spaces. Consequently, when equipped with a convex structure, modular spaces inherit certain properties with uniformly convex Banach spaces. The aim of this chapter is to study the existence of fixed points of monotone G_ρ-nonexpansive mappings and monotone G_ρ-nonexpansive multivalued mappings. The setting are modular spaces that share some properties with reflexive Banach spaces, in particular, uniformly convex modular spaces.

We first need some definitions and properties concerning modular spaces, asymptotic centers and uniform convexities. For more details, the reader is referred to [1, 24, 37, 59, 65, 67, 92].

5.1. Preliminaries

5.1.1 Modular spaces

Definition 5.1.1. Let X be a vector space over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$). A functional $\rho : X \to [0, \infty]$ is called a modular if

(i) $\rho(x) = 0$ if and only if $x = 0$;

(ii) $\rho(\alpha x) = \rho(x)$ for any $\alpha \in \mathbb{K}$ with $|\alpha| = 1$, $x \in X$;

(iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for any $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$, and $x, y \in X$.

We say that the modular ρ is convex if it satisfies the condition

$$\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$$

for any $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$, and $x, y \in X$.

Definition 5.1.2. A modular ρ defines a corresponding modular space, that is, the vector space X_ρ given by

$$X_\rho = \{x \in X : \lim_{\lambda \to 0} \rho(\lambda x) = 0\}.$$

The Luxemburg norm $\| \cdot \|_\rho : X_\rho \to [0, \infty)$ is defined by

$$\|x\|_\rho = \inf \left\{ \alpha > 0 : \rho\left(\frac{x}{\alpha}\right) \leq 1 \right\}$$

for every $x \in X_\rho$.

In this chapter, when referring to convergence with respect to the norm, we are specifically discussing convergence in the Luxemburg norm.

Example 5.1.3. 1) Let $\mathbb{R}^{\mathbb{N}} := \{(x_n)_n : x_n \in \mathbb{R}, \forall n \geq 1\}$, and consider a function defined by

$$\rho((x_n)_n) = \sum_{n=1}^{\infty} |x_n|^n$$

for any $(x_n)_n \in \mathbb{R}^{\mathbb{N}}$. Then ρ is a convex modular.

2) Let X be a vector space over $\mathbb{R}$, $\|\cdot\|$ a normed on X. Since the function $f : [0, \infty) \to [0, \infty)$ defined by $f(t) = t^p$ with $p \in \mathbb{N}^*$ is convex, $\rho(x) = \|x\|^p$ is a convex modular on X.

3) Let μ be the Lebesgue measure on $\mathbb{R}$. A function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing, continuous and satisfies

   i) $\lim\limits_{t\to\infty} \varphi(t) = \infty$, and

   ii) $\varphi(t) = 0$ if and only if $t = 0$.

   The Orlicz modular is defined by

$$\rho(f) = \int_{\mathbb{R}} \varphi(|f(t)|) d\mu(t),$$

   for every measurable real function f on $\mathbb{R}$.

4) Let D be a domain in $\mathbb{R}^n$. The vector space of all real valued, Borel measurable functions defined on D is denoted by $\mathcal{F}(D)$. Take $k \in \mathcal{F}(D)$ such that $k(x) \in [1, \infty]$ for every $x \in D$. Define $D^k_\infty := \{x \in D : k(x) = \infty\}$, and write

$$\rho_k(f) = \int_{D\setminus D^k_\infty} |f(x)|^{k(x)} d\mu + \sup_{x\in D^k_\infty} |f(x)|$$

   for any $f \in \mathcal{F}(D)$. Then the function ρ is a convex and continuous modular on $\mathcal{F}(D)$ (see [73]).

Definition 5.1.4. Let X_ρ be a modular space.

(a) A sequence $(x_n)_n$ in X_ρ is said to be ρ-converging to $x \in X_\rho$ if $\lim\limits_{n\to\infty} \rho(x_n - x) = 0$ (denoted by $x_n \xrightarrow{\rho} x$).

(b) A sequence $(x_n)_n$ in X_ρ is said to be ρ-Cauchy if $\lim\limits_{n,m\to\infty} \rho(x_n - x_m) = 0$.

(c) The modular space X_ρ is ρ-complete if any ρ-Cauchy sequence is ρ-convergent.

(d) A subset $B \subset X_\rho$ is said to be ρ-closed if for any sequence $(x_n)_n \subset B$ with $x_n \xrightarrow{\rho} x$, then $x \in B$. We denotes $\overline{B}^\rho$ the closure of B with respect to ρ.

(e) A subset $B \subset X_\rho$ is called ρ-bounded if $\operatorname{diam}_\rho(B) := \sup\{\rho(x-y) : x, y \in B\}$ is finite, $\operatorname{diam}_\rho(B)$ is called the ρ-diameter of B.

(f) A set $B \subset X_\rho$ is called ρ-compact, if for any sequence $(x_n)_n \subset X_\rho$ there exists a subsequence $(x_{n_k})_k$ and $x \in B$ such that $x_{n_k} \xrightarrow{\rho} x$.

(g) ρ is said to satisfy the Fatou property if $\rho(x-y) \leq \liminf_{n\to\infty} \rho(x-y_n)$ whenever $y_n \xrightarrow{\rho} y$, for any x, y, y_n in X_ρ.

Clearly, if ρ satisfies the Fatou property, then the ρ-balls

$$B_\rho(x, r) := \{y \in X_\rho : \rho(x-y) \leq r\}$$

with $x \in X_\rho$, $r \geq 0$, are ρ-closed.

Definition 5.1.5. Let ρ be a modular defined on a vector space X. We say that ρ satisfies the Δ_2-type condition if there exists $K > 0$ such that

$$\rho(2x) \leq K\rho(x)$$

for any $x \in X_\rho$. The smallest such constant K will be denoted by $\omega(2)$.

By the Δ_2-type condition, it is natural to define the growth function as follows:

Definition 5.1.6. Let ρ be a convex modular satisfying the Δ_2-type condition. We define a growth function ω_ρ by

$$\omega_\rho(t) = \sup\left\{\frac{\rho(tx)}{\rho(x)} : 0 < \rho(x) < \infty\right\} \quad \forall t \in [0, \infty).$$

It is not difficult to prove the following properties of the growth function.

Lemma 5.1.7. *Let ρ be a convex modular satisfying the Δ_2-type condition. Then the growth function ω_ρ has the following properties:*

(a) For every $t \in [0,\infty)$, $\omega_\rho(t) \in [0, \infty)$;

(b) $\omega_\rho : [0,\infty) \to [0,\infty)$ is a convex, strictly increasing function, and so it is continuous;

(c) $\omega_\rho(ab) \leq \omega_\rho(a)\omega_\rho(b)$ for all $a, b \in [0,\infty)$;

(d) $\omega_\rho^{-1}(a)\omega_\rho^{-1}(b) \leq \omega_\rho^{-1}(ab)$ for every $a, b \in [0, \infty)$, where ω_ρ^{-1} is the function inverse of ω_ρ;

(e) $\|x\|_\rho \leq \frac{1}{\omega_\rho^{-1}\left(\frac{1}{\rho(x)}\right)}$ for every $x \in X_\rho \setminus \{0\}$.

An analogue of the following lemma was proved by Benavides, Khamsi, and Samadi [25] for modular function spaces. For the convenience of the reader, we provide the proof using the same technique.

Lemma 5.1.8. *Let ρ be a convex modular satisfying the Δ_2-type condition. Let $(x_n)_n$ be a sequence in X_ρ such that*

$$\rho(x_{n+1} - x_n) \leq \alpha\varepsilon^n, \quad n \geq 1,$$

where α is a positive constant, and $\varepsilon \in (0, 1)$. Then $(x_n)_n$ is Cauchy in $(X_\rho, \|\cdot\|_\rho)$ and ρ-Cauchy.

Proof. We assume that there exists $n_0 \in \mathbb{N}$ such that $\rho(x_{n+1} - x_n) > 0$ for every $n \geq n_0$. It implies that

$$\frac{1}{\alpha \varepsilon^n} \leq \frac{1}{\rho(x_{n+1} - x_n)}, \quad \forall n \geq n_0.$$

By Lemma 5.1.7, we get

$$\omega^{-1}\Big(\frac{1}{\alpha}\Big)\Big(\omega^{-1}\Big(\frac{1}{\varepsilon}\Big)\Big)^n \leq \omega^{-1}\Big(\frac{1}{\rho(x_{n+1} - x_n)}\Big), \quad \forall n \geq n_0.$$

It deduces that

$$\|x_{n+1} - x_n\|_\rho \leq \frac{1}{\omega^{-1}\Big(\frac{1}{\alpha}\Big)} \frac{1}{\Big(\omega^{-1}\Big(\frac{1}{\varepsilon}\Big)\Big)^n}, \quad \forall n \geq n_0.$$

Since $\omega^{-1}\Big(\frac{1}{\varepsilon}\Big) > 1$, $(x_n)_n$ is a Cauchy sequence in $(X_\rho, \|\cdot\|_\rho)$. Note that under Δ_2-type condition, the modular-convergence and norm-convergence coincide. Hence $(x_n)_n$ is also ρ-Cauchy. □

Lemma 5.1.9. *Let ρ be a convex modular satisfying the Δ_2-type condition. Let $(x_n)_n$, $(y_n)_n$ be two sequences in X_ρ. If $\lim_{n\to\infty} \rho(y_n) = 0$, then*

$$\limsup_{n\to\infty} \rho(x_n + y_n) = \limsup_{n\to\infty} \rho(x_n),$$

and

$$\liminf_{n\to\infty} \rho(x_n + y_n) = \liminf_{n\to\infty} \rho(x_n).$$

Proposition 5.1.10. *Let ρ be a modular defined on X. Then ρ-convergence follows from norm convergence in X_ρ. Norm convergence and ρ-convergence are equivalent in X_ρ if and only if the following condition holds: for every sequence $(x_n)_n \subset X_\rho$, if $\lim_{n\to\infty} \rho(x_n) = 0$, then $\lim_{n\to\infty} \rho(2x_n) = 0$.*

Obviously, if ρ satisfies the Δ_2-type condition then norm convergence and ρ-convergence are equivalent in X_ρ.

As in the Banach space setting (see [33, 50, 64, 65, 76]) the method of asymptotic centers plays an important role in proving fixed point theorems for nonexpansive maps. Some definitions and results concerning asymptotic centers demonstrate adaptability to modular spaces in a straightforward manner.

Let ρ be a convex modular defined on X, C be a nonempty ρ-closed ρ-bounded subset of X_ρ, and $(x_n)_n$ a bounded sequence in X_ρ.

Definition 5.1.11. We define

$$\begin{aligned} r_\rho(C, (x_n)_n) &= \inf\{\limsup_{n\to\infty} \rho(x_n - x) : x \in C\}, \\ A_\rho(C, (x_n)_n) &= \{x \in C : \limsup_{n\to\infty} \rho(x_n - x) = r_\rho(C, (x_n))\}. \end{aligned}$$

The number $r_\rho(C, (x_n)_n)$ and the (possible empty) set $A_\rho(C, (x_n)_n)$ are called, respectively, the ρ-asymptotic radius and the ρ-asymptotic center of $(x_n)_n$ in C.

Obviously, if C is convex then the set $A_\rho(C,(x_n)_n)$ is also convex. Furthermore, the set $A_\rho(C,(x_n)_n)$ is nonempty and closed whenever the modular ρ satisfies the Δ_2-type condition and the space X_ρ satisfies property (R_ρ) (see Lemma 5.1.19 ii)).

Definition 5.1.12. The sequence $(x_n)_n$ is said to be regular relative to C if the asymptotic radii of all subsequences of $(x_n)_n$ (relative to C) are the same. If, in addition, $A_\rho(C,(x_{n_k})_k) = A_\rho(C,(x_n)_n)$ for every subsequence $(x_{n_k})_k$ of $(x_n)_n$ we say that $(x_n)_n$ is asymptotically uniform relative to C.

The following lemma can be proved similarly to the work of Goebel and Kirk (Lemma 15.2, [54]) in a Banach space.

Lemma 5.1.13. *Let X_ρ be a complete modular space, C be a nonempty subset in X_ρ, and $(x_n)_n$ be a bounded sequence in X_ρ. Then $(x_n)_n$ contains a subsequence regular relative to C.*

5.1.2 Uniform convexity in modular spaces

Given that X is a vector space, and ρ is a convex modular function, the Luxemburg norm $\|\cdot\|_\rho$ is induced on X_ρ. In fixed point theory for nonexpansive mappings in Banach spaces, the common approach is to assume the uniform convexity of the norm. This assumption implies reflexivity, subsequently ensuring the weak compactness of closed bounded and convex sets. Early investigations addressed the inquiry of whether the normed vector space $(X_\rho, \|\cdot\|_\rho)$ exhibits uniform convexity. As we will explore later, the concept of uniform convexity of modulars, along with the property (R_ρ) representing a counterpart of reflexivity in Banach spaces, provides us with powerful tools for showing fixed-point properties in modular spaces. This issue was extensively explored in Orlicz function spaces (see [92]). Nakano [93] initiated and studied the modular uniform convexity.

Definition 5.1.14 ([68]). Let ρ be a modular, $r > 0$, and $\varepsilon > 0$. For $i = 1, 2$, we define

$$D_i(r,\varepsilon) = \Big\{(x,y) \in (X_\rho)^2 : \rho(x) \le r, \rho(y) \le r, \rho\Big(\frac{x-y}{i}\Big) \ge r\varepsilon\Big\}.$$

If $D_i(r,\varepsilon) \ne \emptyset$, let

$$\delta_i(r,\varepsilon) = \inf\Big\{1 - \frac{1}{r}\rho\Big(\frac{x+y}{2}\Big) : (x,y) \in D_i(r,\varepsilon)\Big\}.$$

If $D_i(r,\varepsilon) = \emptyset$, we set $\delta_i(r,\varepsilon) = 1$.

(a) We say that ρ satisfies (UCi) if for each $r > 0$ and $\varepsilon > 0$, we have $\delta_i(r,\varepsilon) > 0$. Note that for each $r > 0$, $D_i(r,\varepsilon) \ne \emptyset$ for $\varepsilon > 0$ small enough.

(b) We say that ρ satisfies (UUCi) if for each $s \ge 0$ and $\varepsilon > 0$, there exists $\eta_i(s,\varepsilon) > 0$ depending on s and ε such that

$$\delta_i(r,\varepsilon) > \eta_i(s,\varepsilon) > 0$$

for $r > s$.

(c) We say that ρ is strictly convex (SC) if for every $x, y \in X_\rho$ such that $\rho(x) = \rho(y)$ and

$$\rho\Big(\frac{x+y}{2}\Big) = \frac{\rho(x)+\rho(y)}{2},$$

we have $x = y$.

Remark 5.1.15. Observe that

(a) For i = 1,2, the functions $\delta_i(r,\cdot)$ are nondecreasing for every $r > 0$.

(b) For each $x \in X_\rho$, we put

$$\delta'(r,x) = \inf\Big\{1 - \frac{1}{r}\rho\Big(\frac{x}{2}+y\Big) : y \in X_\rho, \rho(y) \le r, \rho(x+y) \le r\Big\}.$$

Then

$$\begin{aligned} \delta_1(r,\varepsilon) &= \inf\{\delta'(r,x) : x \in X_\rho, \rho(x) \ge r\varepsilon\}, \\ \delta_2(r,\varepsilon) &= \inf\{\delta'(r,x) : x \in X_\rho, \rho(\frac{x}{2}) \ge r\varepsilon\}. \end{aligned}$$

Note that if $\delta'(r,x) > 0$ for any $x \in X_\rho \setminus \{0\}$ and $r > 0$, we say that X_ρ is ρ-uniformly convex in every direction (ρ-UCED) (see Definition 2.3, [69]).

(c) $\delta_1(r,\varepsilon) \le \delta_2(r,\varepsilon)$ for $r > 0$ and $\varepsilon > 0$.

The following properties follows easily from Definition 5.1.14.

Proposition 5.1.16 ([68]). *We have the following relations:*

(a) (UUCi) implies (UCi) for i = 1,2;

(b) (UC1) implies (UC2) implies (SC);

(c) (UUC1) implies (UUC2).

By Remark 5.1.15, we have the following results.

Proposition 5.1.17 ([67]). *If ρ satisfies the Δ_2-type condition, then*

(a) (UC1) and (UC2) are equivalent.

(b) (UUC1) and (UUC2) are equivalent.

Example 5.1.18. Consider Example 5.1.3 3).

1) If we choose $\varphi(t) = e^{|t|} - |t| - 1$ or $\varphi(t) = e^{t^2} - 1$, then the Orlicz modular has UC1 property and does not satisfy the Δ_2-type condition (see [69, 91]).

2) If the modular ρ satisfies the Δ_2-type condition, then the Luxemburg norm is (UC1) (see [61, 87, 91]).

Let us begin with some fundamental results for modular spaces with the (UCi) or (UUCi) property.

Lemma 5.1.19 ([1, 26, 68]). *Let ρ be a convex modular defined on X which satisfies the Fatou property. Assume that X_ρ is complete, and ρ is (UUC2). The following properties hold:*

(a) Let C be a nonempty ρ-closed convex subset of X_ρ. Let $x \in X_\rho$ be such that

$$d_\rho(x, C) = \inf\{\rho(x - y) : y \in C\} < \infty.$$

Then there exists a unique point $c \in C$ such that $d_\rho(x, C) = \rho(x - c)$;

(b) X_ρ satisfies property (R_ρ), i.e., for any nonincreasing sequence $(C_n)_n$ of ρ-closed convex nonempty subsets of X_ρ such that $\sup\limits_{n\geq 1} d_\rho(x, C_n) < \infty$ for some $x \in X_\rho$, then $C_\infty = \bigcap\limits_{n\geq 1} C_n$ is nonempty.

Abdou and Khamsi ([1]) showed that property (R_ρ) can be extended to any family of nonempty ρ-bounded ρ-closed convex subsets which satisfies the finite intersection property.

Proposition 5.1.20 ([1, 26, 68]). *Let ρ be a convex modular defined on X. Assume that ρ is (UUC2), and X_ρ is complete. Let Y be a nonempty ρ-bounded ρ-closed convex subset of X_ρ. Let $(Y_i)_{i\in I}$ be a family of nonempty ρ-closed convex subsets of Y such that $\bigcap\limits_{i\in J} Y_i \neq \emptyset$ for any finite subset J of I. Then $\bigcap\limits_{i\in I} Y_i \neq \emptyset$.*

Definition 5.1.21 ([1]). Let ρ be a convex modular on X, C a nonempty subset of X_ρ, and let $(x_n)_n$ be a sequence in X_ρ. The function $\tau_\rho : C \to [0, \infty]$ defined by

$$\tau_\rho(x) = \limsup_{n\to\infty} \rho(x_n - x) \tag{5.1.1}$$

is called a ρ-type function. A sequence $(c_n)_n$ in C is called a minimizing sequence of τ_ρ if $\lim\limits_{n\to\infty} \tau_\rho(c_n) = \inf\limits_{x\in C} \tau_\rho(x)$.

Theorem 5.1.22. *Let ρ be a convex modular satisfying the Δ_2-type condition. Let Y be a ρ-closed ρ-bounded convex subset of X_ρ, and let $(x_n)_n$ be a sequence in X_ρ. Assume that $\tau_\rho : Y \to [0, \infty]$ is the ρ-type function generated by $(x_n)_n$ such that $\tau_0 := \inf\{\tau_\rho(x) : x \in Y\} < \infty$. Then*

(a) τ_ρ is convex, that is, the subset $\{x \in Y : \tau_\rho(x) \leq r\}$ is convex for every $r \geq 0$;

(b) τ_ρ is weakly lower semicontinuous;

(c) Furthermore, if ρ satisfies the Fatou property and (UUC1) property, then there exists a unique point $y \in Y$ such that

$$\tau_\rho(y) = \inf\{\tau_\rho(x) : x \in Y\}.$$

Proof. It is easy to prove (a). Claim (b) is showed in Lemma 3.4 of [26]. Claim (c) is proved in Proposition 3.7 of [1]. □

5.2. Fixed points of monotone G_ρ-nonexpansive mappings in modular spaces

Firstly, we investigate the case of ρ-nonexpansive mappings.

Definition 5.2.1 ([1]). Let ρ be a modular defined on a vector space X and $C \subseteq X_\rho$. A mapping $T : C \to C$ is called ρ-nonexpansive if for every $x, y \in C$,

$$\rho(T(x) - T(y)) \leq \rho(x - y).$$

It is easy to see that a ρ-nonexpansive mapping $T : C \to C$ is ρ-continuous in the meaning that $T(x_n) \overset{\rho}{\to} T(x)$ whenever $x_n \overset{\rho}{\to} x$, $x_n, x \in C$ for any $n \geq 1$. Using the compactness and continuity, Abdou and Khamsi (Theorem 4.4, [1]) established the following theorem.

Theorem 5.2.2 ([1]). *Let X_ρ be a ρ-complete modular space with a convex modular ρ. Let C be a nonempty ρ-compact convex ρ-bounded subset of X_ρ. Then any ρ-nonexpansive mapping $T : C \to C$ has a fixed point.*

The assumption of ρ-compactness is strong. A weaker assumption was considered in the case of uniform convexity. In 2017, Abdou and Khamsi [1] proved a result similar to Browder-Göhde's fixed point theorem (see [28]) for ρ-nonexpansive mappings in modular spaces.

Theorem 5.2.3 ([1]). *Let ρ be a convex modular that satisfies the Fatou property. Let C be a nonempty ρ-closed convex ρ-bounded subset of X_ρ. Let $T : C \to C$ be a ρ-nonexpansive mapping. Assume that ρ is (UUC1). Then T has a fixed point and* $\mathrm{Fix}(T)$ *is ρ-closed, convex.*

We are going to consider counterparts of Theorems 5.2.2 and 5.2.3 for monotone G_ρ-nonexpansive mappings.

Definition 5.2.4 ([1]). Let ρ be a modular defined on a vector space X and $C \subseteq X_\rho$. Let $G = (V(G), E(G))$ be a digraph on X such that $V(G) \subseteq C$. A mapping $T : C \to C$ is called monotone G_ρ-nonexpansive if T is G-monotone and for every $x, y \in C$ with $y \in [x, \to)_G$, we have

$$\rho(T(x) - T(y)) \leq \rho(x - y).$$

Obviously, monotone G_ρ-nonexpansive maps need not be ρ-continuous.

Example 5.2.5. Let ϕ be a nonnegative-valued, increasing, convex function defined on $\mathbb{R}_+$, and there exists $k \geq 1$ such that $\phi(2t) \leq k\phi(t)$ for any $t \in \mathbb{R}_+$. The Orlicz-Birnbaum space L^ϕ is defined by

$$L^\phi = \Big\{x : [0,1] \to \mathbb{R} : \rho_\phi(x) = \int_{[0,1]} \phi(|x(t)|)dt < \infty\Big\}.$$

Take $R > 0$, and write $B_R := \{x \in L^\phi : \rho_\phi(x) \leq R\}$. Fix $\alpha \in (0,1)$. We define a digraph on L^ϕ as follows:

$$(x, y) \in E(G) \Leftrightarrow x(t) \leq y(t) \text{ for almost every } t \in [0, \alpha].$$

Let $F : [0,1] \times [0,1] \times L^\phi \to \mathbb{R}$ be a measurable function in both variables s and t for every $x \in L^\phi$. Assume that F satisfies the following conditions:

i) $|F(t,s,x)| \le h(t,s)+g(t)|x(s)|$ for a.e. $t,s \in [0,1], x \in L^\phi$, where $0 \le g(t) \le M_0 < 2/K$ for $t \in [0,1]$ is integrable, and

$$\int_{[0,1]} \int_{[0,1]} \phi(h(t,s))dtds < +\infty;$$

ii) $F(t,s,\cdot)$ is nondecreasing for a.e. $t,s \in [0,\alpha]$, and

$$0 \le F(t,s,y) - F(t,s,x) \le y(s) - x(s)$$

for a.e. $t,s \in [0,1]$ and any $x,y \in L^\phi$ such that $(x,y) \in E(G)$.

Consider the operator

$$(Tx)(t) = f(t) + \int_{[0,1]} F(t,s,x)ds$$

for every $t \in [0,1]$, where $f \in L^\phi$. We can prove that T is a monotone G_ρ-nonexpansive mapping on B_R for a sufficiently large R.

Using the finite intersection property and applying Theorem 5.2.2 we have the following fixed point theorem for monotone G_ρ-nonexpansive mappings.

Theorem 5.2.6 ([101]). *Let ρ be a modular in X which satisfies Δ_2-type condition, and let G be a digraph on X_ρ. Assume that X_ρ is ρ-complete, and G-intervals along walks are convex and ρ-closed. Let C be a ρ-compact ρ-bounded convex subset of X_ρ and $T : C \to C$ a monotone G_ρ-nonexpansive mapping. If there exists $c \in C$ such that $T(c) \in [c, \to)_G$, then there is $x_0 \in C$ such that $T(x_0) = x_0$.*

Proof. Since ρ satisfies Δ_2-property, ρ-convergence is equivalent to convergence in the space $(X_\rho, \|\cdot\|_\rho)$. It implies that every ρ-compact subset of X_ρ is compact in $(X_\rho, \|\cdot\|_\rho)$. Now Theorem 3.2.5 implies that there exists $s \in C$ such that $[s,s]_G \neq \emptyset$ and $T : [s,s]_G \to [s,s]_G$ is ρ-nonexpansive. It follows from Theorem 5.2.2 that T has a fixed point in $[s,s]_G$. □

In 1980, Bynum [33] introduced normal structure coefficients. Next, Kirk [77] used normal structure in the study of fixed point problems for nonexpansive mappings in Banach spaces. The definitions can be extended to modular spaces. Let C be a nonempty ρ-bounded subset of X_ρ and $x \in X_\rho$. Put

$$\begin{aligned} r_\rho(x,C) &:= \sup\{\rho(x-y) : y \in C\}, \\ r_\rho(C) &:= \inf\{r_\rho(x,C) : x \in \overline{\text{co}}(C)\}. \end{aligned}$$

The number $r_\rho(C)$ is called the ρ-Chebyshev radius of C.

Definition 5.2.7 ([65]). A modular space X_ρ is said to have ρ-normal structure if for any nonempty ρ-bounded ρ-closed convex subset C of X_ρ with $\text{diam}_\rho(C) > 0$, there exists $x \in C$ such that $r_\rho(x,C) < \text{diam}_\rho(C)$.

A modular space X_ρ is said to have ρ-uniform normal structure if there exists a constant $\alpha \in (0,1)$ such that for any subset C as above, there exists $x \in C$ such that $r_\rho(x,C) < \alpha\, \text{diam}_\rho(C)$.

The following result proved by Khamsi [65], is analogous to Kirk's fixed point theorem [77].

Theorem 5.2.8 ([65]). *Let ρ be a convex modular defined on X and satisfies the Fatou property. Assume that X_ρ is ρ-complete modular space. Moreover, X_ρ has the ρ-normal structure and has the property (R_ρ). Let C be a nonempty ρ-bounded ρ-closed convex subset of X_ρ. Then any ρ-nonexpansive mapping $T : C \to C$ has a fixed point.*

Following this direction, we firstly show a sufficient condition for X_ρ to have ρ-normal structure as follows:

Theorem 5.2.9 ([101]). *Let ρ be a modular defined in X. If ρ is (UC2), then X_ρ has ρ-normal structure.*

Proof. Assume that $C \subset X_\rho$ is ρ-closed, convex, ρ-bounded, and $\operatorname{diam}_\rho(C) > 0$. Put $\frac{1}{2}C = \{\frac{c}{2} : c \in C\}$. Then $0 < \operatorname{diam}_\rho(\frac{1}{2}C) \leq \operatorname{diam}_\rho(C)$. Write $d_1 = \operatorname{diam}_\rho(C)$ and $d_2 = \operatorname{diam}_\rho(\frac{1}{2}C)$. Then there are $x, y \in C$ such that $\rho(\frac{x-y}{2}) \geq d_2/2$. For all $z \in C$, we have $\rho(x-z) \leq d_1$ and $\rho(y-z) \leq d_1$. Hence

$$\rho(z-w) \leq d_1 - d_1\delta(d_1, \frac{d_2}{2d_1}),$$

where $w = \frac{x+y}{2}$. Thus

$$r_\rho(w, C) \leq d_1 - d_1\delta(d_1, \frac{d_2}{2d_1}),$$

and since ρ is (UC2), we have $\delta(d_1, \frac{d_2}{2d_1}) > 0$. It follows that $r_\rho(w, C) < d_1$. Therefore, X_ρ has ρ-normal structure. □

From this theorem, we obtain a little improvement on the Abdou-Khamsi's result.

Theorem 5.2.10 ([101]). *Let ρ be a convex modular satisfying the Fatou property and (UUC2). Assume that X_ρ is ρ-complete. Let C be a nonempty ρ-closed convex ρ-bounded subset of X_ρ, and let $T : C \to C$ be a ρ-nonexpansive mapping. Then* $\operatorname{Fix}(T)$ *is a nonempty ρ-closed and convex subset of C.*

Proof. It follows from Proposition 5.1.16 (a) and Theorem 5.2.9 that X_ρ has ρ-normal structure. Furthermore, by Lemma 5.1.19 (b), X_ρ has property (R_ρ). Now Theorem 5.2.8 yields $\operatorname{Fix}(T)$ is nonempty. To prove that $\operatorname{Fix}(T)$ is ρ-closed and convex we can argue in the same way as in [1, Theorem 4.5]. □

By Theorem 5.2.10 we can show a 'modular' version of Browder and Göhde's fixed point theorem for monotone G_ρ-nonexpansive mappings as follows:

Theorem 5.2.11 ([101]). *Let ρ be a convex modular satisfying the Fatou property and (UUC2). Assume that X_ρ is ρ-complete. Let G be a digraph on X_ρ such that G-intervals along walks are convex and ρ-closed. Let C be a nonempty ρ-bounded ρ-closed convex subset of X_ρ and $T : C \to C$ a monotone G_ρ-nonexpansive mapping. If there exists $c \in C$ such that $T(c) \in [c, \to)_G$, then* $\operatorname{Fix}(T)$ *is nonempty.*

Taking a convex modular ρ satisfying the Fatou property and (UUC1), $X_\rho = L_\rho$ and $G :=\leq$, we have Theorem 3.1 of Dehaish-Khamsi [37].

Lemma 5.2.12 ([37]). *Let ρ be a convex modular satisfying the Fatou property and (UUC1), C be a nonempty ρ-bounded ρ-closed convex subset of L_ρ. Let $T : C \to C$ be a ρ-continuous and monotone ρ-nonexpansive mapping. If there exists $f_0 \in C$ such that $f_0 \leq T(f_0)$, then T has a fixed point.*

In a similar way to Theorem 4.2.12, we can prove the existence of common fixed points for a commutative family of monotone G_ρ-nonexpansive mappings.

Theorem 5.2.13 ([101]). *Let ρ be a convex modular satisfying the Fatou property and (UUC2). Assume that X_ρ is ρ-complete. Let G be a digraph on X_ρ such that G-intervals along walks are convex and ρ-closed. Let C be a nonempty ρ-bounded ρ-closed convex subset of X_ρ, and $\mathcal{T}$ a commutative family of monotone G_ρ-nonexpansive mappings from C into C. If there exists $c \in C$ such that $T(c) \in [c, \to)_G$ for every $T \in \mathcal{T}$, then $\bigcap_{T\in\mathcal{T}} \mathrm{Fix}(T)$ is nonempty.*

5.3. Fixed points of monotone G-nonexpansive multivalued mappings in modular spaces

In this section, we formulate some fixed point theorems for monotone G_ρ-nonexpansive multivalued mapping in modular spaces equipped with a digraph.

Let ρ be a modular defined on X, and C be a nonempty subset in X_ρ. We denote by $\mathbb{CL}_\rho(C)$ the collection of all nonempty ρ-closed subsets of C, and by $\mathbb{CP}_\rho(C)$ the collection of all nonempty ρ-compact subsets of C. Assume that $G = (V(G), E(G))$ is a digraph on X_ρ such that $C \subseteq V(G)$.

Definition 5.3.1 ([7]). A multivalued mapping $T : C \to 2^C$ is monotone G_ρ-nonexpansive if for any $x, y \in C$ with $(x, y) \in E(G)$ and any $x_1 \in T(x)$, there exists $y_1 \in T(y)$ such that

$$(x_1, y_1) \in E(G), \quad \text{and} \quad \rho(x_1 - y_1) \leq \rho(x - y).$$

In Definition 5.3.1, if we replace the digraph G with a partial order $\preceq$, which means for any $x, y \in C$ with $x \preceq y$ and any $x_1 \in T(x)$, there exists $y_1 \in T(y)$ such that

$$x_1 \preceq y_1, \quad \text{and} \quad \rho(x_1 - y_1) \leq \rho(x - y),$$

then T is said to be monotone ρ-nonexpansive.

A point $c \in C$ is called a fixed point of T if and only if $c \in T(c)$. The set of all fixed points of a mapping T is denoted by $\mathrm{Fix}(T)$.

Note that in some papers (see [8]), monotone G_ρ-nonexpansive mappings are also referred to as monotone increasing G-nonexpansive mappings.

Example 5.3.2. Consider the vector space $\mathbb{R}^2$ with Euclidean norm $\|(x_1, x_2)\| = \sqrt{x_1^2 + x_2^2}$. By Example 5.1.3, $\rho(x) = \|x\|^2$ for any $x \in \mathbb{R}^2$ is a convex modular. On $\mathbb{R}^2$, we define a digraph $G = (V(G), E(G))$ with $V(G) = \mathbb{R}^2$ and

$$(x, y) \in E(G) \Leftrightarrow x_1 \leq y_1$$

for $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$. Define the mapping $T : \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ by

$$T(x) = \{(z_1, z_2) \in \mathbb{R}^2 : x_1 \leq z_1 \leq x_1 + 1\} \quad \text{for } x = (x_1, x_2) \in \mathbb{R}^2.$$

It is not difficult to prove that T is monotone G_ρ-nonexpansive.

Definition 5.3.3. Let $G = (V(G), E(G))$ be a digraph on a modular space X. We say that the digraph G is convex if for any x_1, x_2, y_1, $y_2 \in V(G)$, and $\lambda \in [0, 1]$, we get

$$(x_1, x_2), (y_1, y_2) \in E(G) \Rightarrow \Big(\lambda x_1 + (1 - \lambda) y_1, \lambda x_2 + (1 - \lambda) y_2\Big) \in E(G).$$

Note that if a transitive digraph G is convex then all G-intervals are convex.

Our first result concerns monotone G_ρ-nonexpansive multivalued mappings $T : C \to \mathbb{CL}_\rho(C)$, where C is G_ρ-compact.

Definition 5.3.4. We say that a nonempty subset C of X_ρ is G_ρ-compact if for any sequence $(x_n)_n$ in C such that $(x_n, x_{n+1}) \in E(G)$ for any $n \geq 1$, there exists a subsequence $(x_{n_k})_k$ of $(x_n)_n$ which ρ-converges to $x \in C$.

In this definition, if we take the graph G to be a partial order $\preceq$, then C is said to be P_ρ-compact.

It is clear that a ρ-compact set is G_ρ-compact, but the converse is not necessarily true.

Example 5.3.5. 1) Consider the set $C = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \leq 1, -1 \leq x_2 \leq 1\}$ with modular function $\rho(x) = x_1^2 + x_2^2$ for $x = (x_1, x_2) \in \mathbb{R}^2$. Obviously, C is ρ-closed. We define a graph G on $\mathbb{R}^2$ by

$$(x, y) \in E(G) \quad \text{if and only if} \quad x_1 \leq y_1$$

for every $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$. Then any sequence $(z_n)_n \subset C$ such that $(z_n, z_{n+1}) \in E(G)$ for $n \geq 1$ satisfies the following conditions:

$$z_n^1 \leq 1, -1 \leq z_n^2 \leq 1 \quad \text{for every } z_n = (z_n^1, z_n^2),$$
$$\text{and} \quad z_n^1 \leq z_{n+1}^1 \quad \text{for every } n \geq 1.$$

It deduces that $(z_n)_n$ is bounded in C. Thus C is G_ρ-compact. It is easy to see that C is not bounded, and hence not ρ-compact.

2) Consider the space $\mathcal{C}(I, \mathbb{R})$ in Example 2.1.11 and the modular function $\rho(x) = \|x\|_{\mathcal{C}}$. A digraph G is generated by partial order, which means

$$(f, g) \in E(G) \quad \Leftrightarrow \quad f(x) \leq g(x) \text{ for all } x \in [0, 1]$$

for every $f, g \in \mathcal{C}(I, \mathbb{R})$. For every $n \geq 1$, put $f_n(x) = x^n$ and $g_n(x) = -(x/2)^n$ for any $x \in I$. Let $C = \{f_n, g_n : n = 1, 2, \ldots\} \cup \{0\}$. Clearly, $C \subset \mathcal{C}(I, \mathbb{R})$. It is not difficult to prove that C is G_ρ-compact and ρ-bounded but it is not ρ-compact.

We have the following lemma for G_ρ-compact sets.

Lemma 5.3.6. *Let X_ρ be a modular space, and C a subset of X_ρ. Assume that G is a transitive digraph in X_ρ such that $C \subseteq V(G)$ and G-intervals are ρ-closed. Let $(x_n)_n$ be a sequence in C satisfying the condition $(x_n, x_{n+1}) \in E(G)$ for any $n \geq 1$. If C is G_ρ-compact, then $\bigcap_{n=1}^{\infty}[x_n, \rightarrow)_G \neq \emptyset$.*

Proof. Since C is G_ρ-compact, there exists a subsequence $(x_{n_k})_k$ of $(x_n)_n$ which ρ-converges to $x \in C$. Since the sequence $(x_{n_k})_k$ is nondecreasing, we have that for each $k \geq 1$, $x_{n_m} \in [x_{n_k}, \rightarrow)_G$ for any $m \geq k$. It follows from the closedness of G-intervals that $x \in [x_{n_k}, \rightarrow)_G$.

On the other hand, for each $n \geq 1$, there exists $k \geq 1$ such that $n_k \geq n$, and thus $(x_n, x_{n_k}) \in E(G)$. It implies that $x \in [x_n, \rightarrow)_G$ for any $n \geq 1$. Therefore, $x \in \bigcap_{n=1}^{\infty}[x_n, \rightarrow)_G$. □

Theorem 5.3.7. *Let ρ be a convex modular defined on X that satisfies the Δ_2-type condition. Assume that X_ρ is complete. Let C be a nonempty G_ρ-compact ρ-bounded convex subset of X_ρ. Assume that G is a transitive and convex digraph in X_ρ, and G-intervals are ρ-closed. Then any monotone G_ρ-nonexpansive multivalued mapping $T : C \to \mathbb{CL}_\rho(C)$ has a fixed point provided $C_\rho = \{x \in C : (x, y) \in E(G) \text{ for some } y \in T(x)\} \neq \emptyset$.*

Proof. Assume that $x_0 \in C_\rho$. Hence there exists $x_0' \in T(x_0)$ such that $(x_0, x_0') \in E(G)$. Since C is convex, we can define the mapping $T_1 : C \to \mathbb{CL}_\rho(C)$ by

$$T_1(x) = \frac{1}{2}x_0 + \frac{1}{2}T(x) \quad \forall x \in C.$$

Since $x_0' \in T(x_0)$, there exists $y_0 \in T_1(x_0)$ such that $y_0 = \frac{1}{2}x_0 + \frac{1}{2}x_0'$. Since G-intervals are convex and $(x_0, x_0') \in E(G)$, we have $(x_0, y_0), (y_0, x_0') \in E(G)$.

Since T is monotone G_ρ-nonexpansive, there exists $y_0' \in T(y_0)$ such that

$$(x_0', y_0') \in E(G) \quad \text{and} \quad \rho(x_0' - y_0') \leq \rho(x_0 - y_0).$$

Put

$$y_1 = \frac{1}{2}x_0 + \frac{1}{2}y_0'.$$

Since $y_0' \in T(y_0)$, $y_1 \in T_1(y_0)$. Clearly, $(x_0, y_1), (y_1, y_0') \in E(G)$. Since G is convex, $(y_0, y_1) \in E(G)$, and

$$\rho(y_1 - y_0) \leq \frac{1}{2}\rho(y_0 - x_0).$$

Since $(y_0, y_1) \in E(G)$, there exists $y_1' \in T(y_1)$ such that

$$(y_0', y_1') \in E(G) \quad \text{and} \quad \rho(y_1' - y_0') \leq \rho(y_1 - y_0).$$

Put

$$y_2 = \frac{1}{2}x_0 + \frac{1}{2}y_1'.$$

Then we have $y_2 \in T_1(y_1)$, $(y_1, y_2) \in E(G)$, and

$$\rho(y_2 - y_1) \leq \frac{1}{2}\rho(y_1 - y_0) \leq \frac{1}{2^2}\rho(y_0 - x_0).$$

By induction we can form two sequence $(y_n)_n$ and $(y'_n)_n$ such that

$$y_{n+1} = \frac{1}{2}x_0 + \frac{1}{2}y'_n,$$

$$y_{n+1} \in T_1(y_n), (y_n, y_{n+1}) \in E(G),$$
$$y'_n \in T(y_n), (y_{n+1}, y'_n), (y'_n, y'_{n+1}) \in E(G) \quad \text{for every } n \geq 0,$$

and

$$\rho(y'_{n+1} - y'_n) \leq \rho(y_{n+1} - y_n) \leq \frac{1}{2}\rho(y_n - y_{n-1}) \quad \text{for every } n \geq 0.$$

Thus we have

$$\rho(y'_{n+1} - y'_n) \leq \frac{1}{2^{n+1}}\rho(y_0 - x_0), \quad \forall n \geq 0.$$

By Lemma 5.1.8, $(y'_n)_n$ is a Cauchy sequence. Since X_ρ is complete, and C is ρ-compact, there exists $t_1 \in C$ such that $\lim_{n\to\infty} \rho(y'_n - t_1) = 0$. Since $([y'_n, \rightarrow)_G)_n$ is nonincreasing in X_ρ, we have as in the proof of Lemma 5.3.6 that $(y'_n, t_1) \in E(G)$ for any $n \geq 0$. Hence $(x_0, t_1) \in E(G)$. Write

$$x_1 = \frac{1}{2}x_0 + \frac{1}{2}t_1.$$

We have

$$\rho(y_{n+1} - x_1) \leq \frac{1}{2}\rho(y'_n - t_1),$$

and thus $\lim_{n\to\infty} \rho(y_n - x_1) = 0$. In the same way, we get $(y_n, x_1) \in E(G)$ for any $n \geq 0$. Then there exists $z_n \in T_1(x_1)$ such that $(y_{n+1}, z_n) \in E(G)$ and

$$\rho(z_n - y_{n+1}) \leq \frac{1}{2}\rho(x_1 - y_n), \ \forall n \geq 0.$$

By Lemma 5.1.9, we have

$$\limsup_{n\to\infty} \rho(z_n - y_{n+1}) = \limsup_{n\to\infty} \rho\Big((z_n - x_1) - (y_{n+1} - x_1)\Big) = \limsup_{n\to\infty} \rho(z_n - x_1).$$

It is easy to prove that the sequence $(z_n)_n$ also converges to x_1. Since $T_1(x_1)$ is ρ-closed, we deduce that $x_1 \in T_1(x_1)$. Note that $(x_0, x_1), (x_1, t_1) \in E(G), t_1 \in T(x_1)$, thus $x_1 \in C_\rho$.

By induction, we can build two sequences $(x_n)_n$, $(t_{n+1})_n$ such that for each $n \geq 0$,

$$x_{n+1} = \frac{1}{n+2}x_n + \Big(1 - \frac{1}{n+2}\Big)t_{n+1},$$

where $t_{n+1} \in T(x_{n+1})$, and x_{n+1} is a fixed point of T_{n+1}, with $T_{n+1} : C \to \mathbb{CL}_\rho(C)$ defined by

$$T_{n+1}(x) = \frac{1}{n+2}x_n + \Big(1 - \frac{1}{n+2}\Big)T(x) \quad \forall x \in C,$$

and $(x_n, x_{n+1}), (t_{n+1}, t_{n+2}) \in E(G)$ for any $n \geq 0$,

Since C is G_ρ-compact, there exists a subsequence $(x_{n_k})_k$ converging to $t \in C$. Since the digraph G is transitive, we have $(x_{n_k}, x_{n_{k+1}}) \in E(G)$ for every $k \geq 0$. It implies that $(x_{n_k}, t) \in E(G)$ for any $k \geq 0$, and since G is transitive, $(x_n, t) \in E(G)$ for any $n \geq 0$.

Moreover we have that

$$\rho(x_{n+1}-t_{n+1}) \leq \frac{1}{n+2}\rho(x_n-t_{n+1}) \leq \frac{1}{n+2}\operatorname{diam}_\rho(C)$$

for any $n \geq 0$. Hence $\lim_{n\to\infty}\rho(x_{n+1}-t_{n+1}) = 0$. By using similar argument as above, the sequence $(t_{n_k})_k$ converges to t.

Since T is monotone G_ρ-nonexpansive and $(x_n, t) \in E(G)$, for every $n \geq 1$ there exists $w_n \in T(t)$ such that $(t_n, w_n) \in E(G)$ and

$$\rho(t_n - w_n) \leq \rho(x_n - t).$$

By Lemma 5.1.9, we deduce that

$$\lim_{k\to\infty}\rho(w_{n_k} - t) = 0.$$

Since $T(t)$ is ρ-closed, we have $t \in T(t)$, i.e., $t \in \operatorname{Fix}(T)$. □

In Theorem 5.3.7, if we equip X with a partial order, we have the following theorem.

Theorem 5.3.8. *Let ρ be a convex modular defined on X, and satisfies the Δ_2-type condition. Assume that X_ρ is equipped with a partial order $\preceq$ such that order intervals are ρ-closed. Let C be a nonempty P_ρ-compact subset of X_ρ. Then any monotone ρ-nonexpansive multivalued mapping $T : C \to \mathbb{CL}_\rho(C)$ has a fixed point provided $C_\rho = \{x \in C : x \preceq y \text{ for some } y \in T(x)\} \neq \emptyset$.*

Proof. Take $x_0 \in C_\rho$. Then there exits $y_0 \in T(x_0)$ such that $x_0 \preceq y_0$. By monotonicity of T, there exists $y_1 \in T(y_0)$ such that

$$y_0 \preceq y_1 \quad \text{and} \quad \rho(y_1 - y_0) \leq \rho(y_0 - x_0).$$

By induction we can form a sequence $(y_n)_n$ such that

$$y_n \preceq y_{n+1} \quad \text{and} \quad \rho(y_{n+1} - y_n) \leq \rho(y_n - y_{n-1}).$$

Since C is P_ρ-compact, there exits a subsequence $(y_{n_k})_k$ such that

$$\lim_{k\to\infty}\rho(y_{n_k} - c) = 0,$$

where $c \in C$. By the closedness of order intervals, we have

$$y_n \preceq c \quad \text{for every } n \geq 0.$$

Assume that t is an upper bound of $(y_n)_n$, then $y_n \in (\leftarrow, t]$ for any $n \geq 0$. Since order intervals are ρ-closed, we get $c \in (\leftarrow, t]$, so that $c \preceq t$. Thus we have $c = \sup_n y_n$.

We are going to prove that $\lim_{n\to\infty}\rho(y_n - c) = 0$. Indeed, assume that $(y_n)_n$ does not ρ-converge to x. Then there exists $\epsilon > 0$ and a subsequence $(y_{n_l})_l$ of $(y_n)_n$ such that

$$y_{n_l} \notin V(c,\epsilon) = \{x \in X_\rho : \rho(x - c) < \epsilon\} \text{ for any } l \geq 1. \tag{5.3.1}$$

Since $(y_{n_l})_l$ is nondecreasing, there exists a subsequence $(y_{n_{l_m}})_m$ of $(y_{n_l})_l$ such that $\lim_{m\to\infty} \rho(y_{n_{l_m}} - c_1) = 0$. In the same way, we have $c_1 = \sup_n y_n$. Thus, $c = c_1$. This contradicts (5.3.1). Therefore, $\lim_{n\to\infty} \rho(y_n - c) = 0$. Since $y_n \preceq c$ for any $n \geq 0$, there exists $z_n \in T(c)$ such that

$$y_{n+1} \preceq z_n, \quad \rho(y_{n+1} - z_n) \leq \rho(y_n - c)$$

By Lemma 5.1.9, we get $\lim_{n\to\infty} \rho(z_n - c) = 0$. Since $T(c)$ is closed, $c \in T(c)$. □

Next we show a 'modular' version of Lim's fixed point theorem (see [85]) for monotone G_ρ-nonexpansive multivalued mappings.

Theorem 5.3.9. *Let ρ be a convex modular defined on X that satisfies the Δ_2-type condition and the Fatou property. Assume that X_ρ is complete, and ρ is (UUC1). Let C be a nonempty ρ-closed ρ-bounded convex subset of X_ρ. Let G be a transitive and convex digraph such that G-intervals are ρ-closed. Let $T : C \to \mathbb{CP}_\rho(C)$ be a monotone G_ρ-nonexpansive multivalued mapping. If there exists $x_0 \in C$ such that $(x_0, y_0) \in E(G)$ for some $y_0 \in T(x_0)$, then T has a fixed point.*

Proof. Using the argument as in Theorem 5.3.7, we can construct two sequences $(x_n)_n$ and $(y_n)_n$ such that $y_n \in T(x_n)$ and

$$x_{n+1} = \frac{1}{n+2} x_n + \Big(1 - \frac{1}{n+2}\Big) y_{n+1},$$

$$(x_n, x_{n+1}), (x_n, y_n), (y_n, y_{n+1}) \in E(G) \text{ for every } n \geq 0,$$

and

$$\lim_{n\to\infty} \rho(x_n - y_n) = 0.$$

Since $(x_n)_n$ is nondecreasing, $\{[x_n, \to)_G,\ n \geq 0\}$ is a nonincreasing sequence of nonempty ρ-bounded ρ-closed convex subsets of X. It follows from Lemma 5.1.19 (b) that

$$C_\infty = \bigcap_{n\geq 0} [x_n, \to)_G \cap C = \bigcap_{n\geq 0} \{x \in C : (x_n, x) \in E(G)\} \neq \emptyset.$$

Now Lemma 5.1.13 implies the existence of a subsequence $(x_{n_k})_k$ of $(x_n)_n$ such that for each subsequence $(x_{n_{k_l}})_l$ of $(x_{n_k})_k$ we have

$$r(C_\infty, (x_{n_{k_l}})) = r(C_\infty, (x_{n_k})).$$

From Theorem 5.1.22 there exists a unique $c \in C_\infty$ such that

$$\limsup_{k\to\infty} \rho(x_{n_k} - c) = \inf\{\limsup_{k\to\infty} \rho(x_{n_k} - x) : x \in C_\infty\} = r(C_\infty, (x_{n_k})).$$

Thus we have $(x_{n_k}, c) \in E(G)$ for any $k \geq 1$. Since T is a monotone G_ρ-nonexpansive multivalued mapping, there exists $c_{n_k} \in T(c)$ such that

$$(y_{n_k}, c_{n_k}) \in E(G) \quad \text{and} \quad \rho(y_{n_k}, c_{n_k}) \leq \rho(x_{n_k}, c)$$

for any $k \geq 1$. Since $T(c)$ is ρ-compact, there exists a subsequence $(c_{n_{k_l}})_l$ of $(c_{n_k})_k$ such that $\lim\limits_{l\to\infty} c_{n_{k_l}} = c' \in T(c)$. First we prove that $c' \in C_\infty$. Indeed, for each $n \geq 1$ and $n_{k_l} \geq n$, we get

$$(x_n, y_n), (y_n, y_{n_{k_l}}), (y_{n_{k_l}}, c_{n_{k_l}}) \in E(G).$$

By the closedness of G-intervals, we deduce that

$$c' \in \bigcap_{n\geq 0} [x_n, \rightarrow)_G.$$

Now, we are going to prove that $c = c'$. We have

$$\begin{aligned}\rho(x_{n_{k_l}} - \frac{c+c'}{2}) &\leq \frac{1}{2}\rho(x_{n_{k_l}} - c) + \frac{1}{2}\rho(x_{n_{k_l}} - c') \\ &= \frac{1}{2}\rho(x_{n_{k_l}} - c) + \frac{1}{2}\rho\Big((x_{n_{k_l}} - y_{n_{k_l}}) + (y_{n_{k_l}} - c_{n_{k_l}}) + (c_{n_{k_l}} - c')\Big).\end{aligned}$$

Taking upper limit as $l \to \infty$ and using Lemma 5.1.9, we get

$$\begin{aligned}\limsup_{l\to\infty} \rho(x_{n_{k_l}} - \frac{c+c'}{2}) &\leq \frac{1}{2}\limsup_{l\to\infty} \rho(x_{n_{k_l}} - c) + \frac{1}{2}\limsup_{l\to\infty} \rho(y_{n_{k_l}} - c_{n_{k_l}}) \\ &\leq \frac{1}{2}\limsup_{l\to\infty} \rho(x_{n_{k_l}} - c) + \frac{1}{2}\limsup_{l\to\infty} \rho(x_{n_{k_l}} - c) \\ &= \limsup_{l\to\infty} \rho(x_{n_{k_l}} - c) \leq \limsup_{k\to\infty} \rho(x_{n_k} - c) = r(C_\infty, (x_{n_k})).\end{aligned}$$

Since C_∞ is nonempty ρ-bounded ρ-closed convex, we have

$$\limsup_{l\to\infty} \rho(x_{n_{k_l}} - \frac{c+c'}{2}) \geq r(C_\infty, (x_{n_{k_l}})).$$

Thus

$$r(C_\infty, (x_{n_k})) = r(C_\infty, (x_{n_{k_l}})) \leq \limsup_{l\to\infty} \rho(x_{n_{k_l}} - \frac{c+c'}{2}) \leq r(C_\infty, (x_{n_k})).$$

Hence

$$r(C_\infty, (x_{n_{k_l}})) = \limsup_{l\to\infty} \rho(x_{n_{k_l}} - \frac{c+c'}{2}) = r(C_\infty, (x_{n_k})). \tag{5.3.2}$$

On the other hand, we have

$$r(C_\infty, (x_{n_{k_l}})) \leq \limsup_{l\to\infty} \rho(x_{n_{k_l}} - c) \leq \limsup_{k\to\infty} \rho(x_{n_k} - c) = r(C_\infty, (x_{n_k})). \tag{5.3.3}$$

Combining 5.3.2 with 5.3.3 and using Theorem 5.1.22 (c), we have $c = (c + c')/2$, i.e., $c = c'$. Therefore, c is a fixed point of T. □

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