Departamento de Análisis Matemático

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Report on Dau Hong Quan's PhD

To the Doctoral School of the Pedagogical University of Krakow in the discipline of Mathematics

The memoir submitted by the PhD Student Mr. Dau Hong Quan is a quite thorough monograph on fixed point results for monotone mappings on ordered metric spaces. The motivation for this memoir seems to be clear: to open the range of known results on existence of fixed points for single and multivalued mappings under monotonicity conditions on abstract spaces. In this way, we find a monograph where we see how some of the most relevant topics on Metric Fixed Point Theory are extended to monotone self-mappings defined on ordered metric spaces and graphs.

This kind of problems, as it is well explained in the memoir by providing a large amount of references, was first addressed by authors familiar to the study of fixed points for mappings under metric conditions as those of being contractive or nonexpansive. Therefore it was natural to them to consider monotone nonexpansive mappings on ordered metric spaces in a plain way and wonder about existence of fixed points for these mappings. However, it was first noticed in [47] that the nonexpansive condition was superfluous in several previous results on Banach spaces when monotonicity was taken under consideration. The author of the memoir takes this fact as starting point of his research to widen, in what seems to be a systematic study, the range of results for which monotonicity, either by itself or added to a very mild nonexpansivity condition, may be enough to ensure existence of fixed points. It is the case that many of the results here presented improve other recent results in the literature.

The memoir is divided into 5 chapters. Chapter 1 is a brief introduction where we find a first description of the approaches and results that are going to find in the work. Chapter 2 is devoted to existence of fixed points for monotone mappings which do not increase the compactness of sets, that is, the author studies here extensions of classical Darbo-Sadovskii theorem for monotone mappings. The problem is split into the Darbo condition and, later, the weaker Sadovskii one. Once the straight case is studied, that is, the case of a single mapping, new results about common fixed points of commutative family of mappings are also provided in a successful way. Next step is to jump from single-valued mappings to multi-valued mappings under Darbo-Sadovskii conditions. This is also addressed in this chapter. For this, a proper understanding of what monotonicity means for multi-valued mappings will be needed. Results obtained bring Darbo-Sadovskii conditions to monotone mappings on ordered metric spaces. Chapter 2 is completed with some applications of the obtained results to the study of integral equations, differential equations and integral inclusions, showing that the range of applicability of these results is much larger than other related results in the literature.

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In Chapter 3 the author introduces digraphs (Definition 3.1.1) as a main object. In fact, now mappings under consideration will by defined on these structures, which basically are directed graph in which we can impose an order. If we have an order, then we can talk about monotonicity of mappings and, if the graph is a subset of a metric space, we can even consider metric conditions. As a result, the author deals with the so-called monotone G-nonexpansive mappings (Section 3.3) in Banach spaces as the main topic in this chapter. First thing that the author explain about these mappings is that, even thought they carried the nonexpansive name on them, they are a kind of weak nonexpansive mappings which need not even be continuous (Example 3.3.2). Therefore, the author will extend many established results on monotone nonexpansive mappings already existing in the literature. In the next section, a new wider class of mappings is introduced and studied. That is the class of monotone G-asymptotically nonexpansive mappings. This section if followed by the studied of common fixed point of a family of commutative G-monotone mappings under different conditions. The chapter is closed with some applications of the obtained results to the study of integral equations. Something important to remark here is that in this chapter we find, in Section 2, what is going to be a key result for a large part that what is to come. This is Theorem 3.2.5, one of the most important in the whole memoir.

In Chapter 4 we find the same main concept of monotone G-nonexpansive mappings as above but now they act on uniformly convex geodesic spaces. The first goal in this chapter is to study geometric properties of geodesic spaces so a robust structure of uniform convexity is well understood. This brings hyperbolic spaces to the work as a particular case of spaces under consideration. As a result, they show that under mild conditions on the modulus of convexity strong enough geometric properties are obtained for the geodesic space. All this work about the geometry of uniformly convex geodesic spaces is covered in the first section of Chapter 4. Then the problem of fixed points of monotone G-nonexpansive mapping on metric and geodesic metric spaces is addressed from Section 2. Both cases, the single and the multi-valued ones, are taken into consideration in a successful way bringing up a whole new collection of interesting results on existence of fixed points for under several situations.

In the last chapter of the memoir, Chapter 5, the author still considers alike kind of problems for monotone mappings but now the ambient spaces are the so-called modular spaces (Definition 5.1.1). Modular spaces are a class of Banach spaces which enjoys of a special mapping, called the modular of the space, that brings nice properties to the space. These spaces have been widely studied by several authors in Metric Fixed Point Property, several references are provided about it in the memoir. In this chapter the class of monotone G-nonexpansive mapping is introduced for modular spaces and, as a result, several new results of existence of fixed points for such mappings in the single and multi-valued case are obtained. These new results are true extensions of other results already existing in the literature and require of a deep work on the properties of modular spaces to be achieved.

In conclusion, this is a very serious research work which should grant the doctoral degree to the author. The whole work can be considered as a monograph on fixed point for monotone mappings on abstract spaces. The approach to the subject that we find here is not only new but full of new insights. The collection of results is impressive due to the quality, the quantity and the significant improvement that

brings to the theory that it deals with. The work has been written in a very careful way and exposition is clear and well-organized. I did not find any serious mistake that should be comment in this report. The list of typos that I noticed is so short that I will not list them.

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