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# Review of the doctoral dissertation of Mr. Dau Hong Quan, M.Sc., entitled "Fixed point theorems in ordered spaces"

The doctoral dissertation of Mr. Quan concerns a problem of the existence of fixed points of monotone mappings acting on a metric space endowed with a partial order or a graph structure. He considers both single-valued as well as multivalued mappings. The dissertation is 85 pages long and is divided into 5 chapters. The bibliography is extensive and contains 120 entries.

A short four-page Chapter 1 is the introduction to the topic. At the beginning of Chapter 1 the author recalls a fundamental fixed point theorem for monotone mappings due to Knaster and Tarski [79]. Let me note that the author of [79] is A. Tarski, not B. Knaster as stated in the bibliography.

On the other hand, Chapter 2 is the longest chapter in the dissertation and has 28 pages. The author presents here several fixed point theorems for monotone mappings on ordered metric spaces using measures of noncompactness. There are some minor inaccuracies in this chapter. In particular, in Example 2.1.5, the author shows that for SOME two elements $z$ and $t$ from a set $Y$ there exists $v \in Y$ such that $z \preceq v$ and $t \preceq V$, from which he deduces that the set $Y$ is directed! On 7th page the author accepts the somewhat risky agreement that a monotonic sequence will continue to be understood as a non-decreasing sequence. Then, however, under this agreement Lemma 2.1.12 part (i) is not true: consider, e.g., the sequence $\left(\frac{1}{n}\right)$. Proposition 2.1.14 is trivial as it stands. Probably, the phrase 'each subsequence' should be replaced by 'some subsequence'.

The main result of Chapter 2 is Theorem 2.2.6 giving sufficient conditions for the existence of fixed points of monotone self-mappings of a nonempty closed bounded subset in a complete ordered metric space. The result is interesting and has a nice proof with the use of the Kuratowski-Zorn lemma. The idea of this proof is also used in the proofs of the following results: Theorem 2.2.10 for a finite commutative family of monotone mappings, Theorem 2.3.4

for a monotone multivalued mapping and Theorem 2.4.5 for a commutative pair of monotone mappings, one of which is single-valued and the second is multivalued. In my opinion a stronger assumption is needed in Theorem 2.2.6: namely, a measure of noncompactness $\nu$ on $X$ should be non-singular, i.e., for any finite subset $A$ of $X$, $\nu(A) = 0$. This assumption is used in a proof of property (vi) given on page 9; more precisely, a proof of the non-emptiness of the set $\Omega_\infty$ depends on it (see, e.g., [113, p. 19] though the authors of [113] assume stronger that $\nu$ is regular). In particular, it suffices that $\nu$ is regular and indeed this assumption appears in Theorems 2.2.10, 2.3.4 and 2.4.5.

Theorems 2.2.7 and 2.2.8 deal with monotone sequentially continuous mappings and in this case it turns out that for any $x_0 \in X$ such that $x_0 \preceq Tx_0$, the sequence $(T^n x_0)$ converges to a fixed point of $T$. Let me note that the proof of Theorem 2.2.8 shows something more: the set $\operatorname{Fix} T$ has not only a minimal element, but also a smallest element. It also has a greatest element, not just a maximal element.

Lemma 2.2.11 should rather be called a theorem or a corollary (to Theorem 2.2.10) as it is interesting itself. I would welcome a discussion on the comparison of this lemma and Theorem 2.2.7. The latter result uses stronger assumptions and has a constructive proof whereas the proof of Lemma 2.2.11 depends on the axiom of choice. It would be interesting to find an example of a mapping as in Lemma 2.2.11 for which sequences $(T^n x_0)$ with $x_0 \preceq Tx_0$ do not converge. Another question: Lemma 2.2.11 was used in the proof of Theorem 2.5.7 on the existence of solutions of a first order initial value problem with discontinuities. Is it possible to apply here Theorem 2.2.7? If this were the case, we would be able to find approximate solutions of the problem.

It seems to me that Example 2.3.6 is not correct. If we consider a sequence $(t_n)$ defined by

$$t_1 := 2 \quad \text{and} \quad t_n := \frac{1}{2} \quad \text{for} \quad n \geqslant 2$$

and we set $e_1 := (1, 0, \ldots)$, then

$$Te_1 = \left\{ (y_n) : \frac{1}{2} \leqslant y_1 \leqslant 2 \quad \text{and} \quad y_n = 0 \quad \text{for} \quad n \geqslant 2 \right\},$$

so $2e_1 \in Te_1$ which contradicts the author's claim that the values of $T$ at elements from the ball $\overline{B}(0, 1)$ are subsets of this ball.

In the proof of Theorem 2.5.1 the author should rather comment on the case where $\gamma = 0$ since in this case we get the equation of incompletely trivial form: $f(x) = F(x, f(x))$.

In Lemma 2.5.6 and Theorem 2.5.7, the author could explain how we understand the integral of a vector function occurring here. Is it the Birkhoff integral?

However, I like this part of Chapter 2 on applications, despite these somewhat critical comments.

In Chapter 3 the author establishes some fixed point theorems for mappings acting on closed bounded convex subsets of a Banach space endowed with a directed graph $G$. The mappings are assumed here to be monotone and nonexpansive or asymptotically nonexpansive with respect to the graph $G$; see Definition 3.3.1. However, a crucial tool here is Theorem 3.2.5 which gives sufficient conditions for the existence of a $G$-interval along walks that is invariant under a $G$-monotone mapping acting on a set of vertices of a directed graph. This result allows to obtain in a relatively simple way several fixed point theorems for monotone $G$-nonexpansive mappings.

Chapter 4 is devoted to the study of uniquely geodesic metric spaces and again to the problem of the existence of fixed points of $G$-monotone nonexpansive mappings. In particular, the author obtains a generalization of the well-known theorem on a convex space in a Hilbert space (see Theorem 4.1.5) and the result stating that a complete uniformly convex metric space has normal structure (see Lemma 4.1.10). Regarding the latter result, let me note that in [101] Mr. Quan together with his co-author established this result under the weaker assumption that a complete metric space is only weakly uniformly convex. Since the dissertation was written later than the article [101] does this mean that this earlier result from [101] is not true? The rest of Chapter 4 contains fixed point theorems for $G$-monotone nonexpansive mappings, including the case of multivalued mappings, and again Theorem 3.2.5 is a useful tool in proving these results.

Finally, in Chapter 5 the author establishes several fixed point theorems for $G$-monotone mappings on modular spaces. Some of them are counterparts of two theorems by Abdou and Khamsi [1] which have been transferred to the case of monotone $G_\rho$-nonexpansive mappings, where $\rho$ denotes a modular. In this chapter the author also explores the concept of uniform convexity of modular. Maybe I missed something but it seems to me that the condition (UUCi) is simply equivalent to the following one: for any $\varepsilon > 0$,

$$\inf\{\delta_i(r,\varepsilon) : r > 0\} > 0$$

since for any $s \geqslant 0$, $\inf\{\delta_i(r,\varepsilon) : r > s\} \geqslant \inf\{\delta_i(r,\varepsilon) : r > 0\}$. Am I right?

On the editorial side, the dissertation is generally well written, although with its relatively large volume it was difficult to avoid some errors. I will mention a few of these, although they do not detract from my entirely positive opinion of this work.

- Page 17, line -10 and page 20, lines -1 and -18: there is 'monotony of $F$' instead of 'monotonicity of $F$'.
- Page 23, line -6: there is 'realation' instead of 'relation'.

- Page 28, line -10: in the definition of the norm in $L^1(I, \mathbb{R})$ there is '$g$' instead of $|g|$.
- Page 30, line 4 and page 31, lines 9 and 10: there is '$F(x, 0)$' instead of '$F(\cdot, 0)$'.
- Page 32, line 13: there is '$F(z, t)$' instead of '$F$'.
- Page 32, line 14: there is '$q(x)$' instead of $q$.
- Page 38, line 4: there is '$\Gamma_X$' instead of '$\Gamma_X(t)$'.
- Page 53, line -17: there is 'monotone $G$-monotone' instead of 'monotone $G$-nonexpansive'.

In conclusion, I think that the doctoral dissertation of Mr. Dau Hong Quan contains many new and interesting results making important contributions to the fixed point theory of mappings acting on spaces with order structure or graph structure. I found the following results particularly interesting: Theorem 2.2.6, Lemma 2.2.11 (which I have suggested to call the theorem), Theorem 2.3.4, Theorem 3.2.5 and Theorem 4.1.5. I also think that one of the strengths of the dissertation is that many applications of the derived fixed point theorems have been given, among others to integral equations of Hammerstein type (Theorem 2.5.1) and of Volterra type (Theorem 2.5.3 and Theorem 3.6.1), an initial value problem with discontinuities (Theorem 2.5.7) and functional integral inclusion (Theorem 2.5.11).

**In my opinion, the doctoral thesis of Mr. Dau Hong Quan meets all the statutory and customary requirements for a doctoral thesis in the discipline of mathematics. I request that the author be admitted to further stages of the procedure for the award of the doctoral degree.**

J. Jachymski