Review of the thesis
Number Structures in the Mathematical Writings of Bernard Bolzano
(Struktury liczbowe w pismach matematycznych Bernarda Bolzana)
by Marlena Fila
under the supervision of dr. hab. Piotr Błasczczyk, prof. UKEN.

I was appointed by the Pedagogical University of Cracow to be a reviewer of the thesis *Struktury liczbowe w pismach matematycznych Bernarda Bolzana* submitted by Marlena Fila for the degree of Doctor of Philosophy at the Pedagogical University of Cracow, Faculty of Exact and Natural Sciences, Department of Mathematics.

The thesis is written in Polish. I translated it with help of DeepL, corrected technical errors in the translation, and compared the problematic parts with the Polish original text. I write the review in English according to the agreement.

1 Introduction

Bernard Bolzano is an eminent Bohemian scholar, who dealt with many fields, mathematics, logic, philosophy, theology, even aesthetics. His work often attracts more interest outside the Czech Republic. That is why I am delighted with the thesis on Bernard Bolzano written in Cracow.

Marlena Fila analyses two the most famous Bolzano mathematical works. First, *A Purely Analytic Proof of the Theorem that between any two Values, which give Results of Opposite Sign, there lies at least one real Root of the equation*, the content fully corresponds to the title. Second, the VII$^{\text{th}}$ section of *Reine Zahlenlehre* that contains the theory of so-called measurable numbers. The goal is to demonstrate that real numbers are treated axiomatically in the former work while in the latter, real numbers are constructed.

2 *A Purely Analytic Proof*

The tiny book from 1817 with only eighteen short paragraphs is a diamond among historical mathematical works. It includes a definition of the continuity of a function, states the Cauchy (or rather Bolzano-Cauchy) criterion, contains proofs of the Cauchy (Bolzano-Cauchy) Theorem, the Supremum Theorem, and the Interemediate Value Theorem which gave the book its title. All theorems and their proofs are analytic that means they are formally proved from basic definitions, without reference to geometry or applied mathematics. It is well known that the proof of the key Cauchy Theorem, which states that any Bolzano-Cauchy sequence has a limit, is incomplete. The reason is obvious. There was no theory of real numbers at the time, so it was impossible to describe a number to which the sequence converges. Bolzano did not prove the *existence* of a limit, he only proved its *uniqueness*.

Marlena Fila describes the structure of *A Purely Analytic Proof*, transcribes the important theorems it contains into the contemporary notation, and explores their connections. One topic of Fila's thesis is the discussion on different definitions of a continuous function, their relationships and Bolzano's use in concrete applications. The above famous theorems and their proofs are described in detail. Particular attention is devoted to the Completeness Theorem, where Fila identifies two incorrect statements, and the Intermediate Value Theorems, where not only Bolzano's proof but also contemporary proofs are given. The validity of this theorem is also considered as one of the definitions of the continuity of function.

The main aim of the first part of Fila's thesis is to show that Bolzano implicitly uses axiomatic theory of real numbers. i.e. of a complete linearly ordered Archimedean field. In the late 18th and early 19th century, algebraic expressions were commonly transformed according to the laws of the commutative field; Bolzano also does it; it can be seen in several cases as Fila pointed out. The ordering is linear, the most problematic being the trichotomy appears in the proof of the Supremum Theorem. Fila verified that the arithmetic operations are congruent to ordering. Bolzano defined the completeness and proved it, as he thought, in the incorrect proof of the Cauchy Theorem. The validity of the Archimedes axiom is referred to in several cases. However, Bolzano applies this axiom as well as the trichotomy axiom only for quantities that denote the length of an interval. They are not valid for ω or Ω which denote „positive quantities that can be as small as we like"

Finally, Fila recapitulates the contents of the preface of *A Purely Analytic Proof*, in which Bolzano presents previous incorrect proofs of the Intermediate Value Theorem. She points out Bolzano's reference to his earlier work *On Mathematical Method*. His deductive method starts with axioms that are intuitive and unprovable, from which we logically prove true theorems. Perhaps this could be the reason why Bolzano does not explicitly state and justify the axioms he uses.

3 *Reine Zahlenlehre*

Reine Zahlenlehre from the early 1830s is a huge Bolzano's project whose aim is to describe the theory of all numbers by the purely analytic method. Its most interesting and most discussed part is the VII$^{\text{th}}$ section. It seems that Bolzano was aware of the missing theory of real numbers and wanted to repair it. The great debate as to whether this is indeed a construction of real numbers has not yet been convincingly concluded. An affirmative answer would mean that Bolzano's construction predates Cantor's and Dedekind's constructions by about forty years. The problem is that there occur some rather minor inconsistencies. Scholars' opinions vary; ranging from condemning the theory as inconsistent, to classifying it as a precursor to Cantor's theory, to believing that one small change in the initial definition can completely correct the entire theory.

Bolzano starts with *infinite number expressions* which are formed by using an infinite number of arithmetic operations. Bolzano selects some of these expressions such that for each natural number q they are elements of a half-open interval $[\frac{p}{q}, \frac{p+1}{q})$ for some natural number p. He calls them *measurable numbers*. The next step is to determine *infinitely small* and *infinitely great* numbers. The last step is the introduction of *equality* and *order*. Two measurable numbers X, Y are *equal* if their difference $|X - Y|$ is infinitely small, X is *greater* than Y if their difference $X - Y$ is positive and not infinitely small.

Besides the well-known Cantor's and Dedekind's construction of real numbers, there is a construction using the non-standard model of rational numbers. We start with the sequences of rational numbers and a non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$. The ultraproduct $\mathbb{Q}^* = \mathbb{Q}^{\mathbb{N}}/\mathcal{F}$ is a linearly ordered non-Archimedean field. Infinitely small and infinitely great elements are defined in the usual way. Limited elements $L_{\mathbb{Q}}$ are those that are not infinitely great. Two limited elements $x, y \in L_{\mathbb{Q}}$ are equivalent $x \doteq y$ if their difference $|x - y|$ is infinitely small. By factorisation we obtain the complete linearly ordered Archimedean field of real numbers, $\Re = L_{\mathbb{Q}}/\doteq$.

This is the construction that Fila uses. She limits herself on infinite sums. This limitation does not lose the generality. Bolzano's infinite number expressions can be represented as sequences of partial *results* and these sequences can be thought of reversibly as representations of infinite *sums*. Considering infinite sums is thus formally sufficient. Infinite sums are interpreted as elements of the ultraproduct $\mathbb{Q}^*$. An infinite sum $a_1 + a_2 + a_3 \ldots in\ inf.$ is interpreted as the class $[(a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots)]_{\mathcal{F}} \in \mathbb{Q}^*$.

The complicate Bolzano's definition of measurable numbers is not necessary now, it is sufficient to interpret measurable numbers as limited elements of the ultrafilter $L_{\mathbb{Q}}$ which form a commutative non-Archimedean ring as well as Bolzano's measurable numbers in this part of the book. Many problems disappear, for instance, the problem with oscillating sequences or an invalid theorem on the sum of measurable numbers. Definitions of infinitely small and infinitely great numbers are analogous. Fila proceeds to go through the book and compares Bolzano's definitions and theorems with her interpretation.

Bolzano's equality corresponds to the equivalence relation $\doteq$ defined on $L_{\mathbb{Q}}$. This is in contrast to the ultraproduct construction, where we factorise modulo equivalence and then work with equivalence classes. Bolzano just defines the equality of measurable numbers and then proceeds to work without distinguishing them from the original measurable numbers. At this point, he loses the infinitesimals, since they are all *equal* to zero, however, he obtains the structure of *real numbers*. In the next paragraphs, Fila shows that Bolzano proved all necessary arithmetic properties of real numbers and their ordering; it is a linearly ordered, dense, and complete Archimedean field; the ordering is congruent with arithmetic operations. The most important property is completeness. Bolzano finally proves in §107 the Completness Theorem, that every Cauchy sequence has a limit that is a measurable number. Consequently, the Supremeum Theorem is valid too.

4 Literature Reviews

A review of studies on these two works by Bolzano is carefully undertaken in both parts of the thesis. Fila has studied many articles and books in German, English, and Polish that have been written about these two works of Bolzano. She deals with their main ideas quite successfully. She also summarizes the content of the previous parts of *Reine Zahlelehre* on natural and rational numbers.

5 Conclusion

Commenting on two famous works by Bolzano and finding a new interpretation of them is a difficult task. Fila carefully researched and analysed both books, studied a lot of materials, and combined them with non-trivial mathematical knowledge. In case of the former, she demonstrated a new reading and in case of the latter, she presented a new interpretation. **Based on the above, she deserves to have her thesis accepted and to receive a Ph.D. in mathematics.**

Prague, March 1, 2024

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