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Fregean fragments of Intuitionistic Propositional
Calculus with mixed congruence type

PhD Dissertation

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Introduction

We are working in the realm of congruence permutable Fregean varieties, following previous research by other authors. Natural examples of such varieties are classes of algebras that form an equivalent algebraic semantic for a deductive system in the sense of Blok and Pigozzi [6]: Boolean algebras, Heyting algebras, or Brouwerian semilattices to name a few. As shown in [14], every such variety has a binary term that behaves like the equivalence connective in the Intuitionistic Propositional Calculus. So, we can find many more examples by taking a variety generated by some reducts of Heyting algebras with equivalence being a member of the reduced language. In particular, one extreme case is the variety generated by equivalence-only reducts; these are the equivalential algebras introduced by Kabziński and Wroński in [13].

The equivalential algebras are also extreme in another way. Any finite congruence algebra is known to be on the spectrum between being locally solvable (so-called type **2**) and congruence distributive (so-called type **3**). Equivalential algebras are on one end of this spectrum, as they represent the type **2**, while Brouwerian semilattices are an example of the type **3**. In our work, we are most interested in the in-between situation, when some parts of the algebra behave differently than other with respect to these properties.

The investigation of such mixed type varieties was started by Przybyło, who, however, worked on the only two such varieties generated by three-element algebras. Because his work took care of the smallest classes, we are turning our attention to the large ones. Our main result is that, if we restrict ourselves to binary terms only, there exist exactly six mixed type congruence permutable Fregean classes that consist of term definable subreducts of Brouwerian semilattices with zero.

Our research started with the goal of extending Przybyło’s results. He considered algebras with two binary operations: equivalence and partial conjunction. One such algebra can be obtained by taking a \mathcal{G} -reduct of a three-element Heyting algebra with the operations in language \mathcal{G} being $(x \rightarrow y) \wedge (y \rightarrow x)$ and $(\neg\neg x) \wedge (\neg\neg y)$. By generating a variety from this algebra, he obtained a Fregean, congruence permutable class with mixed type members. We can obtain a larger such class by taking \mathcal{G} -reducts of all Heyting algebras and generating a variety from them.

In Chapter 2 we show that the class of all \mathcal{G} -subreducts of Heyting algebras forms a variety. The approach we used is based on papers of Słomczyńska [22, 23] in which she worked with other classes of subreducts. The idea is to first “guess” a definition of a variety that contains \mathcal{G} -subreducts, demonstrate some properties of that variety, and gradually show that it is actually equal to the class of subreducts. The second algebra considered by Przybyło is a \mathcal{H} -reduct of a three-element Heyting algebra with \mathcal{H} consisting of $(x \rightarrow y) \wedge (y \rightarrow x)$ and $(\neg\neg x \rightarrow x) \wedge (\neg\neg y \rightarrow y)$. These two binary operations are like before equivalence and partial conjunction, but the conjunction operates on different “part”. Proof that the class of all \mathcal{H} -subreducts forms a variety is also contained in Chapter 2. In both cases the respective varieties are congruence permutable, Fregean and contain mixed type algebras. As a byproduct of the method we used, we also have a finite axiomatization of both classes using identities. The section of Chapter 2 addressing \mathcal{G} -subreducts is a result of a joint work with the advisor, Katarzyna Korwin-Słomczyńska which was already published [18].

As the same approach worked for multiple classes of subreducts, we naturally tried to generalize it. In the beginning of Chapter 3 we discuss what properties are necessary for the method to be applicable. One of assumptions we need is local finiteness of subreducts, this is why later we consider only terms that can be written without the connective “ \vee ”. Moreover, we restricted ourselves to binary terms only. Under these assumptions we were able to represent our research interests as a question about properties of the free Brouwerian semilattice with zero over two generators \mathbf{F}_2 . Because \mathbf{F}_2 has a rather simple representation using a 15-element partially ordered set, we were able to investigate its structure using elementary combinatorics. Chapter 3 constitutes a proof, that there are exactly six classes of

subreducts of Brouwerian semilattices, that we are interested in (congruence permutable, Fregean and containing mixed type algebras). The six classes include the two we investigated in Chapter 2.

In Chapter 4 we discuss the remaining four cases. Language of those classes also contain equivalence and one of the two partial conjunctions, but additionally an unary negation or a certain binary term we denote by “ $-$ ”. Sadly, the particular properties of “ $-$ ” make it less suitable for the approach we used for classes that do not contain this term. This forced us to spend some time on investigating the class of subreducts containing only equivalence and “ $-$ ”. Despite this difficulty, we managed to show that three of four classes are varieties, while the last class of subreducts is only a quasivariety.

For the sake of completeness, Chapter 1 contains the necessary theoretical introduction. It contains the basics of universal algebra, so this dissertation can also be read by a non-specialist. Moreover, it also collects previously known properties of particular algebras and classes that were useful in our research.

Some of the results we obtained can be rephrased in logic terms, for example, the fact that a class of subreducts forms a primitive variety means that the corresponding logic is hereditarily structurally complete. However, in this dissertation, we will focus on the algebraic results only. The results contained in this paper leave some open questions that we are planning to address in the future. We did not show that, if we consider binary terms only, there are no other locally finite congruence permutable Fregean classes of Heyting algebra subreducts. We expect that this question can be solved by analyzing the free Heyting algebra over two generators. The ternary case is also a natural next target, which should be followed up by generalizing the ideas to case with any arity. While researching the mixed case, we also noticed that there is a rich structure of type **2** congruence permutable Fregean classes between the equivalential algebras and equivalential algebras with zero, which we think deserves some attention.

Further research may also address fragments of other logic systems. In particular nuclear Heyting algebras, which form an algebraic semantic for modal intuitionistic logic, should have a richer family of mixed type subreducts. These are Heyt-

ing algebras with an additional unary operation called a nucleus satisfying some compatibility conditions. The nucleus generalizes some properties of the double negation, which should make it possible to introduce more terms that play the part of a partial conjunction.

Chapter 1

Tools and definitions

In this chapter, we provide most of the tools needed to understand the content of this paper. The majority of results in this chapter are folklore or simple corollaries. Any reader familiar with the basics of universal algebra can skip the first three sections. For a more detailed introduction to universal algebra, we refer the reader to the two books [1, 8].

1.1 Basic definitions

Here we have some introductory terminology, which forms a way to generalize the concepts known from groups, fields, lattices, etc. to any arbitrary set with some operations defined on it. The first few definitions might be a little daunting, so for clarity, there is an example at the end of this section which presents what those concepts mean in the language of rings.

Definition 1.1. A *type of algebras* is a set \mathcal{F} so that each member has a non-negative integer assigned, which is called its *arity*. We call members of \mathcal{F} *function symbols*. A symbol of arity 0,1 or 2 is called a *constant*, *unary* or *binary*, respectively. By \mathcal{F}_n we denote the subset of \mathcal{F} consisting of n -ary symbols.

Definition 1.2. An *Algebra of type \mathcal{F}* is a pair $\mathbf{A} = (A, F)$, where A is nonempty set and $F = \{f^{\mathbf{A}} : f \in \mathcal{F}\}$ is a set of functions. Every function in F has codomain A . If a symbol f has arity n , then the domain of $f^{\mathbf{A}}$ is A^n . The image of a tuple

(a_1, \dots, a_n) under $f^{\mathbf{A}}$ is denoted by $f(a_1, \dots, a_n)$. If two algebras are of the same type, then we call them *similar*. For any $n \in \mathbb{N}_0$ an n -ary operation on A is any function $A^n \mapsto A$.

The set A is called an *universe* of the algebra, denoted $A = \text{Univ}(\mathbf{A})$. The elements of F are called *fundamental operations*. If $|A| = 1$ then the algebra is *trivial*, if A is finite then the algebra is *finite*.

Several popular conventions are used for brevity. If the set $\mathcal{F} = \{f_1, \dots, f_n\}$ is finite, we may write $\mathbf{A} = (A, f_1, \dots, f_n)$ instead of $\mathbf{A} = (A, F)$. We may use $x \in \mathbf{A}$ as a shorthand for $x \in \text{Univ}(\mathbf{A})$. If $+$ is some binary symbol, then we can write $a + b$ instead of $+(a, b)$. If we use \cdot as a binary function symbol, we frequently omit it (just like with multiplication) and write ab instead of $a \cdot b$ or $\cdot(a, b)$. By default, we assume associativity to the left, so abc is a shorthand for $(ab)c = \cdot(\cdot(a, b), c)$.

Definition 1.3. For a type of algebras \mathcal{F} and a finite set X we define *set of terms of type \mathcal{F} over X* as the minimal set $T_{\mathcal{F}}(X)$ such that $X \subset T_{\mathcal{F}}(X)$, $\mathcal{F}_0 \subset T_{\mathcal{F}}(X)$ and

$$p_1, \dots, p_n \in T_{\mathcal{F}}(X), f \in \mathcal{F}_n \Rightarrow \text{a tuple } (f, p_1, \dots, p_n) \in T_{\mathcal{F}}(X).$$

One can also construct the above set inductively. Take $T_0 = X \cup \mathcal{F}_0$ and let T_{i+1} be the set of all tuples composed of a n -ary symbol f and n elements from the set $T_0 \cup \dots \cup T_i$, then $T_{\mathcal{F}}(X) = \bigcup T_i$. We call X the *set of variables*. As we will be working with finite X only, we can introduce some ordering on the elements, so $X = \{x_1, \dots, x_{|X|}\}$.

For any algebra $\mathbf{A} = (A, F)$ of type \mathcal{F} and a term $t \in T_{\mathcal{F}}(X)$ we define a function $t^{\mathbf{A}} : A^{|X|} \mapsto A$ in the following way:

- if $t = x_i$ then $t^{\mathbf{A}}(a_1, \dots, a_{|X|}) = a_i$;
- if $t = (f, p_1, \dots, p_n)$ for some $f \in \mathcal{F}_n, p_1, \dots, p_n \in T_{\mathcal{F}}(X)$ then

$$t^{\mathbf{A}}(a_1, \dots, a_{|X|}) = f^{\mathbf{A}}(p_1^{\mathbf{A}}(a_1, \dots, a_{|X|}), \dots, p_n^{\mathbf{A}}(a_1, \dots, a_{|X|})).$$

In particular, for a function symbol $f \in \mathcal{F}_n$ the function $(f, x_1, \dots, x_n)^{\mathbf{A}}$ is the same as the fundamental operation $f^{\mathbf{A}}$.

If it is unambiguous, we can identify the term with the function and write t instead of $t^{\mathbf{A}}$ and $f(p_1, \dots, p_n)$ instead of (f, p_1, \dots, p_n) . If the type is unambiguous, we can also omit it and write $T(X)$ for the set of terms.

Definition 1.4. An *identity* of type \mathcal{F} over X is an expression of a form $p \approx q$ where $p, q \in T_{\mathcal{F}}(X)$ (X is a finite set of variables). We say that \mathbf{A} *satisfies* identity $p \approx q$ if for any $a_1, \dots, a_{|X|} \in A$ we have

$$p^{\mathbf{A}}(a_1, \dots, a_{|X|}) = q^{\mathbf{A}}(a_1, \dots, a_{|X|}).$$

We denote such fact by $\mathbf{A} \models p \approx q$. Of course, this definition is commutative: $\mathbf{A} \models p \approx q \Leftrightarrow \mathbf{A} \models q \approx p$. If a class of algebras \mathcal{K} consists only of algebras that satisfy $p \approx q$, then we can write $\mathcal{K} \models p \approx q$. A class of similar algebras is called an *equational class* if it is defined as all algebras that satisfy a fixed set of identities.

We use “ \approx ” to distinguish identities from equalities. An identity does not mean that the two terms are equal as terms. Rather, the functions they represent are equal on an algebra that satisfies the identity (or on every algebra from a class).

Definition 1.5. Let $\mathbf{A} = (A, F_A), \mathbf{B} = (B, F_B)$ be two algebras of type \mathcal{F} . A function $h : A \mapsto B$ is called a *homomorphism* from \mathbf{A} to \mathbf{B} if for every n -ary $f \in \mathcal{F}$ and elements $x_1, \dots, x_n \in A$ we have

$$h(f^{\mathbf{A}}(x_1, \dots, x_n)) = f^{\mathbf{B}}(h(x_1), \dots, h(x_n)).$$

If h is a homomorphism and a bijection between A and B , then it is called an *isomorphism*. If such an isomorphism exists, then the algebras are said to be *isomorphic*, and we write $\mathbf{A} \simeq \mathbf{B}$. In particular, this means that h^{-1} is a homomorphism from \mathbf{B} to \mathbf{A} .

Definition 1.6. For any algebra $\mathbf{A} = (A, F_A)$ of type \mathcal{F} a nonempty $B \subset A$ is called a *subuniverse* if for every n -ary $f \in F_A$ and elements $b_1, \dots, b_n \in B$ we have $f(b_1, \dots, b_n) \in B$. For any n -ary $f \in \mathcal{F}$ take $f^{\mathbf{B}} = f^{\mathbf{A}}|_{B^n}$ and $F_{\mathbf{B}} = \{f^{\mathbf{B}}, f \in \mathcal{F}\}$. Then $\mathbf{B} = (B, F_{\mathbf{B}})$ is also an algebra of type \mathcal{F} . If this is the case, then we say that \mathbf{B} is a *subalgebra* of \mathbf{A} , and we denote it by $\mathbf{B} \leq \mathbf{A}$. If $C \subset A$ is such that the smallest subuniverse containing C is the whole A , then we say that C *generates* \mathbf{A} .

Remark 1.7. Let h be a homomorphism from \mathbf{A} to \mathbf{B} . An image of any subalgebra of \mathbf{A} under h is a subalgebra of \mathbf{B} . A preimage of any subalgebra of \mathbf{B} is a subalgebra of \mathbf{A} .

Definition 1.8. Take $((A_i, F_i))_{i \in I}$ to be a nonempty indexed family of algebras of type \mathcal{F} . A *direct product* of this family is an algebra $\mathbf{A} = (A, F)$, denoted by $\prod_{i \in I} (A_i, F_i)$, such that its universe $A = \prod_{i \in I} A_i$ is the Cartesian product of universes $A_i, i \in I$. For every n -ary $f \in \mathcal{F}$ and elements $a_1, \dots, a_n \in A$ we define

$$\forall_{i \in I} f^{\mathbf{A}}(a_1, \dots, a_n)(i) = f^{(A_i, F_i)}(a_1(i), \dots, a_n(i)).$$

Direct product is also an algebra of type \mathcal{F} .

Definition 1.9. Let \mathcal{K} be a class of similar algebras. $H(\mathcal{K})$ is the class of all homomorphic images of algebras in \mathcal{K} . $S(\mathcal{K})$ is the class of all subalgebras of algebras in \mathcal{K} . $P(\mathcal{K})$ is the class of all direct products of algebras in \mathcal{K} . By convention we avoid additional parentheses when composing such operators, we write $HS(\mathcal{K})$ instead of $H(S(\mathcal{K}))$.

A *variety* is a class \mathcal{V} of similar algebras such that $H(\mathcal{V}) \subset \mathcal{V}, S(\mathcal{V}) \subset \mathcal{V}, P(\mathcal{V}) \subset \mathcal{V}$.

If an identity holds in a class \mathcal{K} , then it also holds in $H(\mathcal{K}), S(\mathcal{K}), P(\mathcal{K})$. In fact, we have an even stronger relation between identities and varieties.

Theorem 1.10 (Birkhoff's HSP Theorem [3]). *A nonempty class of similar algebras is a variety if and only if it is an equational class.*

Theorem 1.11 (Tarski's V=HSP Theorem [26]). *If \mathcal{K} is a class of similar algebras, then $HSP(\mathcal{K})$ is the smallest variety containing \mathcal{K} .*

Definition 1.12. Let X be a finite set of variables. A *Quasi-identity* of type \mathcal{F} over X is any expression of the form

$$p_1 \approx q_1 \ \& \ \dots \ \& \ p_n \approx q_n \rightsquigarrow p \approx q,$$

where $n \in \mathbb{N}_0, p_1, \dots, p_n, q_1, \dots, q_n, p, q \in T_{\mathcal{F}}(X)$. We say that an algebra \mathbf{A} *satisfies* a quasi-identity if, for any $a_1, \dots, a_{|X|} \in A$, the condition

$$\forall_i p_i^{\mathbf{A}}(a_1, \dots, a_{|X|}) = q_i^{\mathbf{A}}(a_1, \dots, a_{|X|})$$

implies

$$p^{\mathbf{A}}(a_1, \dots, a_{|X|}) = q^{\mathbf{A}}(a_1, \dots, a_{|X|}).$$

If this is the case, we write $\mathbf{A} \models p_1 \approx q_1 \ \& \ \dots \ \& \ p_n \approx q_n \rightsquigarrow p \approx q$. If all algebras in a class \mathcal{Q} satisfy the quasi-identity, we write $\mathcal{Q} \models p_1 \approx q_1 \ \& \ \dots \ \& \ p_n \approx q_n \rightsquigarrow p \approx q$. Every identity is a quasi-identity with $n = 0$. A class of similar algebras defined as all algebras satisfying a fixed set of quasi-identities is called a *quasivariety*. In general, quasi-identities are not preserved by homomorphisms.

We use the unconventional notation of squiggly arrow “ \rightsquigarrow ” to distinguish quasi-identities from both logical implication “ \Rightarrow ” and a function symbol “ \rightarrow ”. Those three arrows ultimately represent similar concepts, but are slightly different on a technical level. Quasi-identity does not represent a logical implication unless it is interpreted in an algebra or a class of algebras.

Definition 1.13. A variety \mathcal{V} is called *primitive* if each subclass $\mathcal{Q} \subset \mathcal{V}$ that is a quasivariety is also a variety.

Definition 1.14. Let $\mathbf{A} = (A, F_A), \mathbf{B} = (B, F_B)$ be algebras from a class \mathcal{K} and $C \subset A$. We say that \mathbf{A} has a *universal mapping property for \mathcal{K} over C* if, for any function $f : C \mapsto B$, there exists exactly one homomorphism $h : A \mapsto B$ with $h|_C = f$. Moreover if C generates \mathbf{A} then we call \mathbf{A} a *free algebra over C in \mathcal{K}* .

Theorem 1.15. *Let \mathcal{K} be a class of algebras. If \mathbf{A} is a free algebra over C in \mathcal{K} , \mathbf{B} is a free algebra over D in \mathcal{K} and $|C| = |D|$, then $\mathbf{A} \simeq \mathbf{B}$.*

Definition 1.16. A variety \mathcal{V} is called *locally finite* if, for any finite set C , the free algebra over C in \mathcal{V} is also finite. Equivalently, for any $\mathbf{A} \in \mathcal{V}$ and finite $C \subset \text{Univ}(\mathbf{A})$, the smallest subalgebra of \mathbf{A} which contains C is finite.

Definition 1.17. A *clone* on a nonempty set A is a set of operations (of possibly varying arity) on A , which is closed under composition and contains projections (the functions defined by $f(x_1, \dots, x_n) = x_i$ for some n, i). If $\mathbf{A} = (A, F)$ is an algebra, then $\text{Clo}(\mathbf{A})$ is its *clone of terms* - the smallest clone on A containing the fundamental operations. By $\text{Clo}_n(\mathbf{A})$ we denote its subset of $\text{Clo}(\mathbf{A})$ consisting

of n -ary operations. The *clone of polynomials* of an algebra is the smallest clone containing all fundamental operations and all constants. Two algebras with the same universe are *term equivalent* (*polynomially equivalent*) if they have the same clone of terms (polynomials).

For any algebra \mathbf{A} of type \mathcal{F} and a set $\mathcal{G} \subset \mathcal{F}$ we can obtain an algebra of type \mathcal{G} by simply restricting the set of fundamental operations on \mathbf{A} . Such an object is called a \mathcal{G} -reduct of \mathbf{A} . The following definition generalizes this notion, so that elements of \mathcal{G} can be not only fundamental operations but also terms.

Definition 1.18. Let \mathcal{F} be a type of algebras and $X = \{x_1, \dots, x_n\}$ be a finite set of variables. Let \mathcal{G} be a subset of $T_{\mathcal{F}}(X)$. For any $\mathbf{A} = (A, F)$ of type \mathcal{F} we define its \mathcal{G} -reduct: the algebra $\mathbf{B} = (A, G)$ with the same universe and operations defined by $\forall_{g \in \mathcal{G}} g^{\mathbf{B}} = g^{\mathbf{A}}$ (terms in \mathbf{A} become fundamental operations in \mathbf{B}). If we further have $\mathbf{C} \leq \mathbf{B}$ (a subalgebra of type \mathcal{G}) then we say that \mathbf{C} is a \mathcal{G} -subreduct of \mathbf{A} . If $D \subset A$ and D is closed under all operations of the form $g^{\mathbf{A}}, g \in \mathcal{G}$, then we can take \mathcal{G} -reduct of \mathbf{A} and then restrict universe to D obtaining a subreduct which we simply denote (D, G) .

Theorem 1.19 ([19] Corrolary 5, p.216). *Let \mathcal{Q} be a quasivariety of type \mathcal{F} and X be a finite set of variables. Let $\mathcal{G} \subset T_{\mathcal{F}}(\{X\})$ (so we can define \mathcal{G} -reducts). The class of all \mathcal{G} -subreducts of algebras from \mathcal{Q} is a quasivariety.*

Remark 1.20. Assuming in the above Theorem that \mathcal{Q} is an arbitrary variety is not enough to guarantee that the class of subreducts is a variety. However, if we assume \mathcal{Q} is a primitive variety, then as a consequence the class of subreducts must be variety.

The following example assumes that the reader is familiar with the “classic” definition of rings. It tries to capture similarities and differences between the classic and universal approach.

Example 1.21. Let $\mathcal{F} = \{+, \cdot, -, 0, 1\}$ be a type where $+, \cdot$ are binary, $-$ is unary, and $0, 1$ are constants. Any ring with 1 can be considered an algebra of this type.

Let $X = \{x, y, z\}$ be the set of variables. Examples of terms belonging to $T_{\mathcal{F}}(X)$ include:

$$0; x; (+, x, y); (\cdot, z, x); (+, x, (\cdot, (-, y), z)); (+, (+, x, y), x); (+, x, (+, y, x)).$$

Each of them is just a particular way of representing the expressions

$$0; x; x + y; zx; x + ((-y)z); (x + y) + x; x + (y + x),$$

respectively. Each of the above expressions can be interpreted in a ring as a function by treating x, y, z as placeholders for the arguments.

A ring can be defined as an algebra of type \mathcal{F} satisfying a set of conditions and each of those conditions can be written as an identity. For example, $(xy)z \approx x(yz)$ is the associativity of multiplication and $x + (-x) \approx 0$ is the existence of an additive inverse. Notice that instead of an existential quantifier we just have a unary function that assigns the inverse to each element. As rings are the class of algebras of type \mathcal{F} satisfying a fixed set of identities, they form an equational class and hence a variety. The “classic” notions of ring homomorphism, subring and product of rings coincide with the “universal” ones.

Fields do not form the variety, as product of fields need not to be a field (in fact, they are not even a quasivariety). Take $\mathcal{G} = \{+, -, 1, 0\}$, then \mathcal{G} -subreduct of a ring is an abelian group with an additional distinguished element 1. Denote the class of all such subreducts \mathcal{K} . If \mathbf{A} is a ring in which $0 = 1$, then for any $a \in A, a = 1 \cdot a = 0 \cdot a = 0$, so the ring is trivial. So the quasi-identity

$$0 \approx 1 \rightsquigarrow x = 0$$

holds in rings and it must also hold in \mathcal{K} . Take \mathbf{C} to be the \mathcal{G} -reduct of the ring \mathbb{Z}_2 . $\mathbf{C} \times \mathbf{C}$ has a nontrivial homomorphic image in which $0 = 1$ (take a function that maps $0 = (0, 0), 1 = (1, 1)$ to one element and $(0, 1), (1, 0)$ to another). Hence, $HSP(\mathcal{K}) \not\subseteq \mathcal{K}$, which shows that \mathcal{K} is not a variety. This is exactly because we cannot express $0 \neq 1$ with a set of identities without using \cdot .

1.2 Examples of varieties

Here we collect information about some classes of algebras that are common in universal algebra and would be useful later. Namely (semi)lattices, Boolean groups, Boolean algebras, and Heyting algebras.

Definition 1.22. An algebra (L, \wedge) with one binary operation is called a *semilattice* if the following identities hold

$$x \wedge y \approx y \wedge x \quad (\wedge \text{ is commutative}),$$

$$x \wedge (y \wedge z) \approx (x \wedge y) \wedge z \quad (\wedge \text{ is associative}),$$

$$x \approx x \wedge x \quad (\wedge \text{ is idempotent}).$$

Definition 1.23. An algebra $\mathbf{L} = (L, \wedge, \vee)$ with two binary operations is called a *lattice* if the following identities hold

$$x \wedge y \approx y \wedge x \quad (\wedge \text{ is commutative}),$$

$$x \vee y \approx y \vee x \quad (\vee \text{ is commutative}),$$

$$x \wedge (y \wedge z) \approx (x \wedge y) \wedge z \quad (\wedge \text{ is associative}),$$

$$x \vee (y \vee z) \approx (x \vee y) \vee z \quad (\vee \text{ is associative}),$$

$$x \approx x \vee (x \wedge y) \approx x \wedge (x \vee y) \quad (\text{the absorption law}).$$

A subset $X \subset L$ is *upward closed* if $\forall_{y \in L} \forall_{x \in X} x \leq y \Rightarrow y \in X$. The family of all upward closed sets is denoted $\text{Up}(\mathbf{L})$.

In many sources a reader can find the definition of a lattice which requires also the idempotent identities $x \approx x \wedge x, x \approx x \vee x$. However, those two identities are consequences of the six listed, and they are usually added in order to clarify that $(L, \wedge), (L, \vee)$ are semilattices. The operations \wedge, \vee are frequently called conjunction and disjunction, or “meet” and “join”. Each lattice has an associated partial ordering of its elements defined by $x \leq y \Leftrightarrow x \vee y = y \Leftrightarrow x \wedge y = x$. Conversely, any partial ordering can be represented as a lattice as long as every pair of elements has a well-defined least upper bound and greatest lower bound. We can define operations from such an ordering by $x \vee y = \sup\{x, y\}, x \wedge y = \inf\{x, y\}$.

Definition 1.24. An algebra $(L, \wedge, \vee, 1, 0)$ with two binary constants is called a *bounded lattice* if (L, \wedge, \vee) is a lattice and the identities $x \wedge 0 \approx 0, x \vee 1 \approx 1$ hold.

One could also define a bounded lattice to be a lattice with a maximum and a minimum element in the associated ordering. However, doing it that way will force us to use an existential quantifier which would make it impossible to write the definition as an equational class. That's why we use a "trick" of extending the language to contain symbols for those elements.

Definition 1.25. A lattice is *distributive* if it satisfies

$$x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z).$$

Equivalently, one can define distributivity by requiring the dual identity

$$x \vee (y \wedge z) \approx (x \vee y) \wedge (x \vee z).$$

A lattice is *modular* if it satisfies

$$(x \wedge y) \vee (y \wedge z) \approx y \wedge ((x \wedge y) \vee z),$$

or, equivalently, if it satisfies a quasi-identity

$$x \wedge y \approx x \rightsquigarrow x \vee (y \wedge z) \approx y \wedge (x \vee z).$$

Theorem 1.26 ([10, 2]). *Every distributive lattice is modular, but not every modular lattice is distributive. There are two five-element lattices (called N_5 and M_3) that can be used to characterize modularity and distributivity. If a lattice has a sublattice isomorphic to either of those, then it is not distributive; if it has a sublattice isomorphic to N_5 , then it is not even modular (cf. Figure 1.2).*

Definition 1.27. If for every subset X of a lattice \mathbf{L} there exists a least upper bound $\bigvee X$ and a greatest lower bound $\bigwedge X$ in the associated partial ordering, then we say that the lattice is *complete*.

An element y of a complete lattice \mathbf{L} is called *compact* if, for any $X \subset \text{Univ } \mathbf{L}$ such that $y \leq \bigvee X$, there must exist a finite $Y \subset X$ such that $y \leq \bigvee Y$.

A complete lattice is *algebraic* if each element is a least upper bound of some set of compact elements.

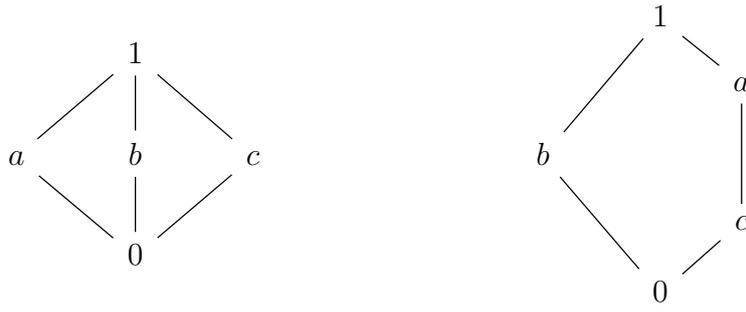


Figure 1.1: The minimal non-distributive lattices M_3 (left) and N_5 (right). N_5 is also non-modular.

Definition 1.28. An element x of a complete lattice \mathbf{L} is *completely meet-irreducible* if for every set $Y \subset \text{Univ } \mathbf{L}$ equality $x = \bigwedge Y$ implies $x \in Y$ (element x cannot be reduced to a meet of other elements). For every completely meet-irreducible x there exists exactly one element x^+ called its *cover* such that

$$\forall_{y \in \text{Univ } \mathbf{L}} x \leq y \Rightarrow (x = y \text{ or } x^+ \leq y).$$

One can similarly define *completely join-irreducible* elements.

Remark 1.29. In any complete lattice $\mathbf{L} = (L, \wedge, \vee)$ there exists an upper bound $1 = \bigvee \text{Univ } \mathbf{L}$ and the lower bound $0 = \bigwedge \text{Univ } \mathbf{L}$. This means that \mathbf{L} is a (\wedge, \vee) -reduct of a bounded lattice. The upper bound 1 is not completely meet-irreducible, because $1 = \bigwedge \emptyset$.

Definition 1.30. Algebra $(G, \cdot, ^{-1}, 1)$ with one binary, one unary and one constant operation is called a *group* if it satisfies the following identities

$$\begin{aligned} (xy)z &\approx x(yz), \\ x1 &\approx x, \\ 1x &\approx x, \\ xx^{-1} &\approx 1, \\ x^{-1}x &\approx 1. \end{aligned}$$

If it also satisfies $xy \approx yx$, then we say that it is *Abelian* or *commutative*. A group that satisfies $x^{-1} \approx x$ is a *Boolean group*.

Remark 1.31. Each Boolean group is commutative.

Unlike in many sources, we defined $^{-1}$ as an unary operation. This, again, is a way to avoid using the existential quantifier, so we can treat groups like any other variety. Of course, we can equivalently define groups to use the additive language $(+, -, 0)$. Boolean groups can also be identified with vector spaces over the two-element field with the binary operation being vector addition.

Definition 1.32. An algebra $(B, \wedge, \vee, \neg, 1, 0)$ is called a *Boolean algebra* if its reduct $(B, \wedge, \vee, 1, 0)$ is a bounded distributive lattice and \neg is a unary operation satisfying $x \wedge \neg x \approx 0, x \vee \neg x \approx 1$.

Boolean algebras are the algebraic counterpart of classical logic; there is a natural mapping between tautologies of logic and identities that hold in all Boolean algebras.

Definition 1.33. An algebra $(H, \wedge, \vee, \rightarrow, 1, 0)$ is called a *Heyting algebra* if its reduct $(H, \wedge, \vee, 1, 0)$ is a bounded distributive lattice; and for any $x, y \in H$, the element $x \rightarrow y$ is a *relative pseudo-complement*: the greatest element z , such that $z \wedge x \leq y$. Heyting algebras form an equational class because the properties of \rightarrow can be rewritten using four identities.

$$\text{He1. } x \rightarrow x \approx 1,$$

$$\text{He2. } (x \rightarrow y) \wedge y \approx y,$$

$$\text{He3. } (x \rightarrow y) \wedge x \approx x \wedge y,$$

$$\text{He4. } x \rightarrow (y \wedge z) \approx (x \rightarrow y) \wedge (x \rightarrow z).$$

An important property of \rightarrow , which can be derived from the above identities, is $x \rightarrow y = 1 \Leftrightarrow x \leq y$.

For simplicity we also use introduce names for some term operations in Heyting algebras: equivalence $x \leftrightarrow y = (x \rightarrow y) \wedge (y \rightarrow x)$, negation $\neg x = x \leftrightarrow 0$, regularization $r_{\mathcal{H}}(x) = \neg\neg x$, densification $d_{\mathcal{H}}(x) = r_{\mathcal{H}}(x) \rightarrow x$. Negation can also be defined as $\neg x = x \rightarrow 0$ and those definitions are equivalent. Elements satisfying $r_{\mathcal{H}}(x) = x$ (or equivalently $d_{\mathcal{H}}(x) = 1$) are called *regular* and those satisfying $d_{\mathcal{H}}(x) = x$ (or

equivalently $r_{\mathcal{H}}(x) = 1$) are *dense*. Every element x can be split into its dense and regular part, because $x = d_{\mathcal{H}}(x) \wedge r_{\mathcal{H}}(x) = d_{\mathcal{H}}(x) \leftrightarrow r_{\mathcal{H}}(x)$. The class of all Heyting algebras is denoted \mathcal{H} .

Heyting algebras constitute an algebraic semantic for intuitionistic propositional calculus, which means that identities which are true in \mathcal{H} correspond to IPC tautologies. As the topic has been thoroughly investigated over the years, there are many tools and ways to verify whether an identity is true in \mathcal{H} . There is even proof assistant software available, for example [17, 9]. Due to a multitude and the availability of possible methods, we will sometimes omit proofs that identities are true in \mathcal{H} , as doing otherwise would cause us to deviate from the subject without providing any significant value.

Remark 1.34. Let $\mathbf{H} = (H, \wedge, \vee, \rightarrow, 1, 0)$ be a Heyting algebra. The function $r_{\mathcal{H}}$ preserves $\wedge, \rightarrow, 1, 0$, so $\text{Im}(r_{\mathcal{H}})$ is a subalgebra of $(H, \wedge, \rightarrow, 1, 0)$. Moreover, the algebra $\mathbf{B} = (r_{\mathcal{H}}(H), \wedge, \vee^{\mathbf{B}}, \neg^{\mathbf{B}}, 1, 0)$ with $x \vee^{\mathbf{B}} y = r_{\mathcal{H}}(x \vee y)$, $\neg^{\mathbf{B}} r_{\mathcal{H}}(x) = r_{\mathcal{H}}(x \rightarrow 0)$ is a Boolean algebra. In particular, $r_{\mathcal{H}}(x \rightarrow y) = (\neg^{\mathbf{B}} r_{\mathcal{H}}(x)) \vee^{\mathbf{B}} r_{\mathcal{H}}(y)$, $r_{\mathcal{H}}(x \leftrightarrow y) = r_{\mathcal{H}}(x \wedge y) \vee^{\mathbf{B}} ((\neg^{\mathbf{B}} r_{\mathcal{H}}(x)) \wedge (\neg^{\mathbf{B}} r_{\mathcal{H}}(y)))$.

1.3 Congruences

The congruences are a way of generalizing the notions of a normal subgroup and an ideal of a ring. In principle, a congruence is “an object we can divide by” to obtain a quotient algebra. Many known properties are still true in this abstract setting. For additional clarity, we again add a more familiar example at the end of this section.

Definition 1.35. Let $\mathbf{A} = (A, F)$ be an arbitrary algebra. An equivalence relation θ on the set A is called a *congruence*, if for any n -ary $f \in F$ and elements $a_1, \dots, a_n, b_1, \dots, b_n \in A$ such that $(a_1, b_1), \dots, (a_n, b_n) \in \theta$ we have

$$(f(a_1, \dots, a_n), f(b_1, \dots, b_n)) \in \theta.$$

The congruences of a given algebra can be ordered by inclusion (they are subsets of $A \times A$). This ordering has an associated lattice, called the *lattice of congruences*

$\text{Con}(\mathbf{A})$, in which \wedge is a set intersection. If $\text{Con}(\mathbf{A})$ is distributive (respectively, modular) then we say that \mathbf{A} is *congruence distributive* (*congruence modular*). The upper bound in this lattice is the full relation $1_{\mathbf{A}}$; the lower bound is the identity relation $0_{\mathbf{A}}$. A *principal* congruence is the smallest congruence $\Theta(a, b)$ that contains the pair (a, b) . For simplicity, we can write $(a, b) \in \theta$ as $a \equiv_{\theta} b$.

For two binary relations $\alpha, \beta \subset A \times A$, we can define their *relational product* $\alpha \circ \beta = \{(x, z) \in A \times A : \exists y \in A (x, y) \in \alpha, (y, z) \in \beta\}$. If for every $\alpha, \beta \in \text{Con}(\mathbf{A})$ we have $\alpha \circ \beta = \beta \circ \alpha$, then we say that \mathbf{A} is *congruence permutable*. Using the relational product, one can describe \vee operation on congruences (and on equivalence relations). For any $\alpha, \beta \in \text{Con}(\mathbf{A})$ we have

$$\alpha \vee \beta = \alpha \cup (\alpha \circ \beta) \cup (\alpha \circ \beta \circ \alpha) \cup \dots$$

With congruence permutability, the above can be simplified to $\alpha \vee \beta = \alpha \circ \beta$. The lattice of congruences is a sublattice of the lattice of all equivalence relations.

Remark 1.36. If $\theta \in \text{Con}(\mathbf{A})$, and $(a_1, b_1), \dots, (a_n, b_n) \in \theta$, then

$$(f(a_1, \dots, a_n), f(b_1, \dots, b_n)) \in \theta$$

holds for arbitrary $f \in \text{Clo}_n(\mathbf{A})$, not only for fundamental operations. If \mathbf{B} is a reduct of \mathbf{A} , then $\text{Clo}(\mathbf{B}) \subset \text{Clo}(\mathbf{A})$ and $\text{Con}(\mathbf{A}) \subset \text{Con}(\mathbf{B})$.

Theorem 1.37 ([5, 11]). *The lattice of congruences of any algebra is an algebraic lattice. Every algebraic lattice is isomorphic to the congruence lattice of some algebra.*

The above fact explains the name “algebraic lattice”. We can use the duality between algebraic lattices and congruences to rewrite [4, Theorem 2] as a theorem about algebraic lattices.

Theorem 1.38. *Every element x of an algebraic lattice \mathbf{L} can be represented as $x = \bigwedge Y$ for some set of completely meet-irreducible $Y \subset \text{Univ } \mathbf{L}$.*

Theorem 1.39. *If an algebra \mathbf{A} is congruence permutable, then it is congruence modular.*

Theorem 1.40 ([20]). *A variety \mathcal{V} is congruence permutable if and only if there exists a ternary term (so-called **Malcev term**) p such that*

$$\mathcal{V} \models p(y, y, x) \approx p(x, y, y) \approx x.$$

The complete characterization of congruence distributivity using terms is slightly more complicated and can be found in [15]. Using that characterization, one can obtain a sufficient condition, which would be enough for our needs.

Corollary 1.41 ([15], use Theorem 2.1 with $n = 2$). *If a variety has a ternary term (so-called **majority term**) M such that*

$$\mathcal{V} \models M(x, x, y) \approx M(x, y, x) \approx M(y, x, x) \approx x$$

then it is congruence distributive.

Definition 1.42. Let \mathbf{A} be an arbitrary algebra of type \mathcal{F} and $\theta \in \text{Con}(\mathbf{A})$. We define a *quotient algebra* \mathbf{A}/θ with universe consisting of equivalence classes of θ and for any n -ary $f \in \mathcal{F}$ and elements a_1, \dots, a_n we take

$$f^{\mathbf{A}/\theta}(a_1/\theta, \dots, a_n/\theta) = f^{\mathbf{A}}(a_1, \dots, a_n).$$

This is well defined exactly because θ is a congruence.

Definition 1.43. For any map $h : \mathbf{A} \mapsto \mathbf{B}$, we define its *kernel* $\ker h = \{(x, y) \in \text{Univ}(\mathbf{A}) \times \text{Univ}(\mathbf{A}) : h(x) = h(y)\}$.

Theorem 1.44 (Homomorphism Theorem). *For any homomorphism $h : \mathbf{A} \mapsto \mathbf{B}$, its kernel is a congruence. An image of such homomorphism is a subalgebra of \mathbf{B} isomorphic to $\mathbf{A}/\ker h$.*

Definition 1.45. Let \mathbf{A} be an arbitrary algebra and $\theta, \phi \in \text{Con}(\mathbf{A}), \theta \subset \phi$. We define

$$\phi/\theta = \{(x/\theta, y/\theta) : (x, y) \in \phi\}.$$

Such a set is a congruence on \mathbf{A}/θ .

Theorem 1.46 (Second Isomorphism Theorem). *Let \mathbf{A} be an arbitrary algebra. If $\theta, \phi \in \text{Con}(\mathbf{A}), \theta \subset \phi$, then $\mathbf{A}/\phi \simeq (\mathbf{A}/\theta)/(\phi/\theta)$.*

Theorem 1.47 (Correspondence Theorem). *Let \mathbf{A} be an arbitrary algebra and $\alpha \in \text{Con}(\mathbf{A})$. The mapping $h : \{\theta \in \text{Con}(\mathbf{A}) : \alpha \subset \theta \subset 1_{\mathbf{A}}\} \mapsto \text{Con}(\mathbf{A}/\alpha)$ defined by $h(\theta) = \theta/\alpha$ is a lattice isomorphism.*

Lemma 1.48. Let \mathbf{A} be an arbitrary algebra. For any $\alpha \in \text{Con}(\mathbf{A})$ and elements $(a, b \in A$ we have $(\Theta_{\mathbf{A}}(a, b) \vee \alpha)/\alpha = \Theta_{\mathbf{A}/\alpha}(a/\alpha, b/\alpha)$.

Proof. As $(a, b) \in \Theta_{\mathbf{A}}(a, b)$, then by definition $(a/\alpha, b/\alpha) \in (\Theta_{\mathbf{A}}(a, b) \vee \alpha)/\alpha$, so $\Theta_{\mathbf{A}/\alpha}(a/\alpha, b/\alpha) \subset (\Theta_{\mathbf{A}}(a, b) \vee \alpha)/\alpha$. On the other hand, from the previous Theorem there exists $\theta \in \text{Con}(\mathbf{A})$ such that $\alpha \subset \theta$ and $\theta/\alpha = \Theta_{\mathbf{A}/\alpha}(a/\alpha, b/\alpha)$. We have $(a/\alpha, b/\alpha) \in \theta/\alpha$, hence $(a, b) \in \theta$, $\Theta_{\mathbf{A}}(a, b) \subset \theta$ and $(\Theta_{\mathbf{A}}(a, b) \vee \alpha)/\alpha \subset \theta/\alpha = \Theta_{\mathbf{A}/\alpha}(a/\alpha, b/\alpha)$. \square

Sometimes, it is useful to decompose an algebra into smaller parts that are easier to work with. For example, any algebra that is a direct product can be decomposed into components. However, by generalizing the notion of a product, one can reduce an algebra to even simpler components.

Definition 1.49. An algebra \mathbf{A} is *subdirectly irreducible* if $\text{Con}(\mathbf{A}) \setminus \{0_{\mathbf{A}}\}$ has a minimal element. This element is called a *monolith* of \mathbf{A} and is usually denoted by $\mu_{\mathbf{A}}$ or just μ .

An algebra \mathbf{A} is *simple* if $\text{Con}(\mathbf{A})$ has two elements.

Definition 1.50. Let $(A_i, F_i)_{i \in I}$ be an indexed family of similar algebras. Let (A, F) be a subalgebra of $\prod_{i \in I} (A_i, F_i)$. If for any $i \in I$ the image of A under the projection $(a_1, \dots, a_n) \mapsto a_i$ is A_i , then (A, F) is a *subdirect product* of that family.

Theorem 1.51. *An algebra \mathbf{A} is subdirectly irreducible if and only if for any family $(A_i, F_i)_{i \in I}$ such that $\mathbf{A} \simeq \prod_{i \in I} (A_i, F_i)$, there exists $i \in I$, $\mathbf{A} \simeq (A_i, F_i)$.*

Remark 1.52. By the Theorem 1.47 a congruence $\theta \in \text{Con}(\mathbf{A})$ is completely meet-irreducible if and only if \mathbf{A}/θ is subdirectly irreducible. If this is the case, θ^+/θ is the monolith of \mathbf{A}/θ .

Definition 1.53. The set of all completely meet-irreducible congruences of an algebra \mathbf{A} is denoted by $\text{Cm}(\mathbf{A})$.

Theorem 1.54 ([4]). *Every algebra \mathbf{A} is isomorphic to a subdirect product of (subdirectly irreducible) algebras $\{\mathbf{A}/\theta_i\}_{i \in I}$, where $\forall i \in I \theta_i \in \text{Cm}(\mathbf{A})$.*

A simple corollary to the above theorem is the fact that different varieties must have different subdirectly irreducible algebras.

Example 1.55. Let $\mathbf{G} = (G, \cdot, ^{-1}, 1)$ be a group. A congruence on \mathbf{G} is an equivalence relation $\theta \subset G \times G$ such that its equivalence classes are cosets of some normal subgroup $\mathbf{H} < \mathbf{G}$ (and $\mathbf{H} = 1/\theta$). The quotient algebra in the universal sense \mathbf{G}/θ is isomorphic to the quotient group \mathbf{G}/\mathbf{H} . This one-to-one correspondence between congruences and normal subgroups is, in fact, a lattice isomorphism. \wedge of congruences corresponds to intersection of normal subgroups. In group theory, a kernel of a homomorphism h is the coset $1/h$, because this set uniquely identifies the whole congruence $((x, y) \in \theta \Leftrightarrow h(x) = h(y) \Leftrightarrow xy^{-1} \in H \Leftrightarrow (xy^{-1})/\theta = 1/\theta)$; we say that the groups are 1-regular.

The term $p(x, y, z) = xy^{-1}z$ is a Malcev term, so groups are congruence permutable, and hence congruence modular. The group $\mathbb{Z}_2 \times \mathbb{Z}_2$ has a congruence lattice isomorphic to M_3 (the minimal non-distributive lattice), so the groups are not congruence distributive. A subdirectly irreducible group has a nontrivial normal subgroup that is contained in all nontrivial normal subgroups. A simple group has only itself and the trivial group as a normal subgroup; equivalently, it has only two congruences: full and identity relations.

1.4 Fregean algebras

This paper is mostly concerned with classes of algebras with a common constant term 1, such that for any algebra \mathbf{A} in that class the following holds:

- 1-regularity: congruences are determined by the equivalence classes of 1,

$$\alpha, \beta \in \text{Con}(\mathbf{A}), 1/\alpha = 1/\beta \Rightarrow \alpha = \beta.$$

- Congruence orderability: the assignment $x \mapsto \Theta_{\mathbf{A}}(1, x)$ is injective.

Definition 1.56 ([14]). Algebra is called *Fregean* if it is 1-regular and congruence orderable. A (quasi-)variety of algebras is called *Fregean* if all algebras in that class are Fregean (with respect to the same constant).

The congruence orderability implies that the relation $x \leq y \Leftrightarrow \Theta(1, x) \supset \Theta(1, y)$ is antisymmetric, so we can use it to order the elements.

Definition 1.57. In a Fregean algebra *natural ordering* is the partial ordering defined by

$$x \leq y \Leftrightarrow \Theta(1, x) \supset \Theta(1, y).$$

The constant 1 is the greatest element in this order.

Many well-known varieties are Fregean: Heyting algebras, Boolean algebras, Brouwerian semilattices, etc. Another important class is equivalential algebras, which arise as subreducts of Heyting algebras with the only operation being the equivalence (defined by $x \leftrightarrow y = (x \rightarrow y) \wedge (y \rightarrow x)$).

Definition 1.58 ([13]). An algebra (A, \cdot) with one binary operation is called an *equivalential algebra* if it satisfies the following identities (by convention we associate to the left so $xyzw$ means $((xy)z)w$):

E1. $xyx \approx y$;

E2. $xyzx \approx xz(yz)$;

E3. $xy(xzx)(xzx) \approx xy$.

A *filter* on equivalential algebra \mathbf{A} is a nonempty set $F \subset \text{Univ}(\mathbf{A})$ such that for any $x, y \in \mathbf{A}$ we have $x \in F \Rightarrow xy \in F$ and $x, xy \in F \Rightarrow y \in F$. The variety of all equivalential algebras is denoted by \mathcal{E} .

The above definition implies many other simple yet useful properties of equivalential algebras, detailed proofs can be found in [13].

Corollary 1.59. *The equivalential algebras are congruence permutable and Fregean with respect to the term $1 := xx$. In an equivalential algebra $x = y \Leftrightarrow xy = 1$ and the following hold:*

E4. $xx \approx yy$;

E5. $xy \approx yx$;

E6. $xyyy \approx xy$;

E7. $xyzzy \approx x(yz)y$;

E8. $xyyzz \approx xzzyy$;

E9. $xyy(yz)(yz) \approx xyzzz$;

E10. $xyzzz \approx xzz(yzz)$.

There is a one-to-one mapping between congruences and filters of an equivalential algebra. For any congruence θ the set $1/\theta$ is a filter; for any filter F the set $\{(x, y) : xy \in F\}$ is a congruence.

There is a particular property of equivalential algebras that makes it somewhat easier to manipulate congruences.

Definition 1.60. Let \mathbf{A} be any algebra with constant term 1 and \mathbf{e} be a binary term on that algebra. We say that \mathbf{e} is a *principal congruence term* if the condition $\Theta_{\mathbf{A}}(a, b) = \Theta_{\mathbf{A}}(1, \mathbf{e}(a, b))$ is satisfied for any $a, b \in \mathbf{A}$.

Theorem 1.61 ([14, Theorem 3.8, Corrolary 3.9]). *In every congruence permutable Fregean variety \mathcal{V} there exists a term \leftrightarrow such that for any algebra $\mathbf{A} \in \mathcal{V}$ we have $(\text{Univ}(\mathbf{A}), \leftrightarrow) \in \mathcal{E}$. The term \leftrightarrow is the principal congruence term for all $\mathbf{A} \in \mathcal{V}$.*

The above means that the class \mathcal{E} of equivalential algebras is in a way the most natural variety of congruence permutable Fregean varieties. Of course, in the case of equivalential algebras \cdot is the principal congruence term.

Now we want to sum up some known results regarding Fregean algebras that we will use in the paper.

Lemma 1.62 ([14, Lemma 2.1]). *If a congruence orderable (with respect to the constant 1) algebra \mathbf{A} is subdirectly irreducible with the monolith μ , then $1/\mu$ has two elements and all other equivalence classes of μ are one-element.*

Proof. As $\mu \neq 0_{\mathbf{A}}$, the class $1/\mu$ must contain some element $a \neq 1$. This means that $\Theta(1, a) \subset \mu$, but on the other hand $\Theta(1, a) \neq 0_{\mathbf{A}}$. Because the monolith is the second smallest congruence we must have $\Theta(1, a) = \mu$. If there is another element $b \in 1/\mu$ that is not 1, then by the same reasoning $\Theta(1, b) = \mu = \Theta(1, a)$, so by congruence orderability $a = b$. Hence $1/\mu$ has exactly two elements, it remains to show that it is the only non-trivial class.

Assume that we have two elements $(a, b) \in \mu$ and $a, b \notin 1/\mu$. The congruence $\Theta(1, a)$ must be larger than μ , so $(a, b) \in \Theta(1, a)$ and, by transitivity, $(1, b) \in \Theta(1, a)$. Similarly $(1, a) \in \Theta(1, b)$, so $\Theta(1, b) = \Theta(1, a)$, which by congruence orderability implies $a = b$. \square

In fact, the above property of a monolith can be used to characterize congruence orderable varieties [14, Corollary 2.2]. The lemma implies that in every subdirectly irreducible congruence orderable algebra there exists a second-largest element.

Definition 1.63. Let \mathbf{A} be subdirectly irreducible with the monolith μ and congruence orderable with respect to term 1. We use $*_{\mathbf{A}}$ to denote the unique element $*_{\mathbf{A}} \in \mathbf{A}, *_{\mathbf{A}} \neq 1$ such that $(1, *_{\mathbf{A}}) \in \mu$. If the algebra in question is unambiguous, we can omit the subscript and simply write $*$.

Corollary 1.64. *An algebra from a Fregean variety is subdirectly irreducible if and only if it has the second-largest element. Simple algebras in a Fregean variety are two-element.*

Proof. If a Fregean algebra \mathcal{A} is subdirectly irreducible, then from the previous lemma it has an element $*$ that is second-largest. For the converse, assume that $a \in \mathbf{A}$ is second-largest. By 1-regularity, for any $\gamma \in \text{Con}(\mathcal{A}), \gamma \neq 1_{\mathcal{A}}$ the set $1/\gamma$ contains an element $b \neq 1$. But, by the definition of ordering, $\Theta_{\mathbf{A}}(1, a) \subset \Theta_{\mathbf{A}}(1, b) \subset \gamma$, so $\Theta_{\mathbf{A}}(1, a)$ is a monolith.

If \mathbf{B} is simple, then it is subdirectly irreducible. The monolith of \mathbf{B} is neither $1_{\mathbf{B}}$ nor $0_{\mathbf{B}}$ if the universe has more than two elements. \square

Theorem 1.65 ([24, Theorem 4.1, 4.2, 5.2, 5.3]). *Let \mathcal{V} be a congruence permutable Fregean variety of finite type \mathcal{F} such that for every operation $t \in \mathcal{F}_n$ the condition*

$$\mathcal{V} \models t(x_1yy, x_2yy, \dots, x_nyy) \approx t(x_1, x_2, \dots, x_n)yy \quad (\dagger)$$

holds, then

- in every subdirectly irreducible $\mathbf{A} \in \mathcal{V}$ with $|A| > 2$ the set $A \setminus \{*\mathbf{A}\}$ is a subuniverse;
- \mathcal{V} is locally finite;
- if moreover $\{1\}$ is a subuniverse of every algebra in \mathcal{V} , then \mathcal{V} is primitive.

There is only one equivalential algebra with two elements (the reduct of the two-element Boolean algebra), and by removing the smaller element from the universe, we get the trivial algebra. Together with the above theorem, this implies that

Corollary 1.66. *If \mathbf{A} is a subdirectly irreducible equivalential algebra then the set $\text{Univ}(\mathbf{A}) \setminus \{*\}$ is its subuniverse. The variety of equivalential algebras is primitive.*

Corollary 1.67. *In a subdirectly irreducible equivalential algebra \mathbf{A} the identity $x* = x$ holds for any $x \notin \{1, *\}$. Moreover, if $xy = *$ for some $x, y \in \mathbf{A}$, then $\{x, y\} = \{1, *\}$.*

Proof. For any x in a subdirectly irreducible equivalential algebra we have $x = x1 \equiv_{\mu} x*$, so by Lemma 1.62 either $x = x*$ or $\{x, x*\} = \{1, *\}$. The second possibility obviously cannot occur if we assume $x \notin \{1, *\}$. If $xy = *$, then $\Theta(1, xy) = \mu$, so $\Theta(x, y) = \mu$ and, by Lemma 1.62, $\{x, y\} = \{1, *\}$. \square

Lemma 1.68. Let $\mathbf{A} = (A, F)$ be a subdirectly irreducible algebra from a Fregean variety. If $A \setminus \{*\}$ is a subuniverse, then the subalgebra \mathbf{B} with such a universe is isomorphic to $\mathbf{A}/\mu_{\mathbf{A}}$.

Proof. Take the map $h : A \mapsto A \setminus \{*\}$ given by

$$h(x) = \begin{cases} 1 & \text{when } x = *, \\ x & \text{otherwise.} \end{cases}$$

Obviously $\ker h = \mu_{\mathbf{A}}$, so if it is a homomorphism, then $\mathbf{A}/\mu_{\mathbf{A}} \simeq \mathbf{A}/\ker h \simeq \mathbf{B}$. Suppose it is not a homomorphism, then for some n -ary $f \in F$ there exist elements $x_1, \dots, x_n \in A$ such that

$$h(t(x_1, \dots, x_n)) \neq t(h(x_1), \dots, h(x_n)).$$

We know that $h(t(x_1, \dots, x_n)) \equiv_\mu t(x_1, \dots, x_n) \equiv_\mu t(h(x_1), \dots, h(x_n))$ and μ has only one nontrivial equivalence class, $\{1, *\}$. Therefore, the assumed inequality is only possible with one side equal to 1 and the other equal to $*$. The left-hand side cannot be $*$ because it is not in the image of h , so the right-hand side must be equal to $*$. By $A \setminus \{*\}$ being a subuniverse, one of $h(x_1), \dots, h(x_n)$ must be $*$. This is not possible because $*$ is not in the image of h . \square

Definition 1.69. Let \mathbf{A} be an algebra from a congruence modular variety. Given three congruences $\alpha, \beta, \gamma \in \text{Con}(\mathbf{A})$ we say that α *centralizes β modulo γ* if for any $n \in \mathbf{Z}_+, t \in \text{Clo}_{n+1} \mathbf{A}, (a, b) \in \alpha, (c_1, d_1), \dots, (c_n, d_n) \in \beta$ we have

$$\begin{aligned} t(a, c_1, \dots, c_n) \equiv_\gamma t(a, d_1, \dots, d_n) &\Leftrightarrow \\ t(b, c_1, \dots, c_n) \equiv_\gamma t(b, d_1, \dots, d_n). \end{aligned}$$

Commutator of congruences α, β is the smallest congruence $[\alpha, \beta]$ such that α centralizes β modulo $[\alpha, \beta]$.

Remark 1.70. If α centralizes β both modulo γ_1 and modulo γ_2 then it also centralizes modulo $\gamma_1 \wedge \gamma_2$. Any two congruences on a given algebra centralize each other modulo the full congruence. This shows that the commutator is well defined.

Theorem 1.71 ([14, Corollary 2.8]). *Let \mathbf{A} be a finite algebra from a congruence permutable Fregean variety \mathcal{V} . The clone of polynomials of \mathbf{A} is determined by the structure $\text{Concom}(\mathbf{A}) = (\text{Con}(\mathbf{A}); \wedge, \vee, [\cdot, \cdot])$, i.e., the congruence lattice together with the commutator operation. (This means that if $\mathbf{A}, \mathbf{B} \in \mathcal{V}$ have the same universe and $\text{Concom}(\mathbf{A}) \simeq \text{Concom}(\mathbf{B})$ then they are polynomially equivalent).*

The elements of $\text{Concom}(\mathbf{A})$ are binary relations on $\text{Univ}(\mathbf{A})$. As there are only finitely many binary relations on a finite set, there is a simple corollary.

Corollary 1.72 ([14, Corollary 9.2]). *For any $n \in \mathbb{N}_{\geq 1}$ there are only finitely many (up to isomorphism) polynomially nonequivalent algebras \mathbf{A} with n -element universe such that $\mathcal{V}(\mathbf{A})$ is a congruence permutable Fregean variety.*

One of the tools which can be used to investigate finite algebras is Tame Congruence Theory described in [12]. It distinguishes five ways a clone of polynomials can

behave in a “local” sense. These are called types and are denoted **1**, **2**, **3**, **4**, **5**. We do not want to cover this quite wide theory, so we will only focus on what is needed to explain our main research goal. We on purpose avoid defining what a type even is, as explaining the necessary theory would significantly lengthen this chapter. We will be working with congruence permutable algebras, which guarantees that the only possible types are **2** and **3** ([12, Theorem 9.14.]). Fortunately, those two can be easily distinguished using the commutator and a few facts from the mentioned book (mostly Theorem 5.7, 9.10, 4.17).

Corollary 1.73. *Let \mathbf{A} be a finite congruence permutable algebra. If for every $\alpha \in \text{Con}(\mathbf{A})$ we have $[\alpha, \alpha] = \alpha$, then the algebra has type **3**. If for every $\alpha \in \text{Con}(\mathbf{A}), \alpha \neq 0_{\mathbf{A}}$ we have $[\alpha, \alpha] < \alpha$, then the algebra has type **2**. If neither is the case, then the algebra is of mixed type; it has subalgebras of both types.*

*If \mathbf{A} is congruence distributive, then it is of type **3**. An algebra is of type **3** or mixed if and only if there exist $a, b \in \mathbf{A}$ and two binary polynomials p, q such that $(\{a, b\}, p, q)$ is a two-element lattice. It follows that if \mathbf{A} is of type **2**, then all its subalgebras are also of type **2**.*

The above properties will be enough to characterize the types in this dissertation. Because symbols **2**, **3** are also commonly used to describe certain algebras, we will explicitly mention when they are used as types.

1.5 Properties of selected Fregean varieties

We will now investigate properties of some Fregean varieties that we will find useful later. The following statements about Heyting algebras are folklore.

Theorem 1.74. *Let $\mathbf{H} = (H, \wedge, \vee, \rightarrow, 1, 0)$ be a Heyting algebra. The equivalence operation \leftrightarrow is a principal congruence term. For any element $a \in H$ we can describe the principal congruence by*

$$\Theta_{\mathbf{H}}(1, a) = \{(x, y) : a \leq (x \leftrightarrow y)\}.$$

Heyting algebras are congruence orderable.

Theorem 1.75. *The class of Heyting algebras \mathcal{H} is a congruence permutable Fregean variety. The Fregean order in Heyting algebras coincides with the lattice order.*

Lemma 1.76. Let \mathbf{H} be a Heyting algebra. We can extend its universe by adding an element $*_H$ such that, for any $x \in H \setminus \{1\}$, in the lattice order we would have $x < *_H < 1$. The relative pseudocomplement can be also extended in a way that the resulting algebra will be Heyting. The algebra we would obtain in the process is denoted by \mathbf{H}^\oplus , it is subdirectly irreducible.

For any subdirectly irreducible $\mathbf{A} \in \mathcal{H}$, $\mathbf{A}/\mu_{\mathbf{A}} \simeq \mathbf{H}$ if and only if $\mathbf{A} \simeq \mathbf{H}^\oplus$.

We would like to have a result similar to the Lemma 1.76 for other classes of algebras.

Definition 1.77. Let \mathcal{V} be a variety, and $\mathbf{A} = (A, F) \in \mathcal{V}$. If there is up to isomorphism only one subdirectly irreducible algebra $\mathbf{B} \in \mathcal{V}$ such that $\mathbf{B}/\mu_{\mathbf{B}} \simeq \mathbf{A}$ we say that \mathbf{A} has *unique extension (in \mathcal{V})*. We denote such an algebra \mathbf{B} by \mathbf{A}^\oplus .

Theorem 1.78. *Every $\mathbf{A} \in \mathcal{E}$ has a unique extension.*

Proof. Let $\mathbf{A} = (A, \cdot) \in \mathcal{E}$. We define an algebra \mathcal{A}^\oplus with an universe $A \cup \{*\}$ (with $* \notin A$ being a new element) and the operation \cdot satisfying

$$x \cdot^{\mathbf{A}^\oplus} y = \begin{cases} x \cdot^{\mathbf{A}} y & \text{if } x, y \in A; \\ 1 & \text{if } x = y = *; \\ x & \text{if } x \in A, y = *; \\ y & \text{if } y \in A, x = *. \end{cases}$$

By direct checking, such an algebra \mathbf{A}^\oplus is equivalential. For any congruence θ and element $x \in A \setminus \{1\}$ we have

$$x \equiv_\theta 1 \Rightarrow 1 \equiv_\theta x = x \cdot^{\mathbf{A}^\oplus} * \equiv_\theta *,$$

so $*$ is the second-largest in the Fregean order. Corollary 1.64 implies that \mathbf{A}^\oplus is subdirectly irreducible.

To show uniqueness, take any subdirectly irreducible $\mathbf{B} \in \mathcal{E}$, such that $\mathbf{B}/\mu_{\mathbf{B}} \simeq \mathbf{A}$. By Lemma 1.68, \mathbf{A} is a subalgebra of \mathbf{B} obtained by removing $*_{\mathbf{B}}$. From

Corollary 1.67 we have $x \cdot^{\mathbf{B}} *_\mathbf{B} = *_\mathbf{B} \cdot^{\mathbf{B}} x = x$ for any $x \notin \{1, *_\mathbf{B}\}$, and obviously $*_\mathbf{B} \cdot^{\mathbf{B}} *_\mathbf{B} = 1, 1 \cdot^{\mathbf{B}} *_\mathbf{B} = *_\mathbf{B} \cdot^{\mathbf{B}} 1 = *_\mathbf{B}$. It follows that $\mathbf{B} \simeq \mathbf{A}^\oplus$. \square

Definition 1.79. We denote by $\mathbf{2}$ the equivalential algebra with two elements (it is unique up to isomorphism). Using the unique extensions property, we can define $\mathbf{3} = \mathbf{2}^\oplus, \mathbf{4} = \mathbf{3}^\oplus, \dots$. Those algebras (and their isomorphic images) are referred to as *chains*. Each of them can be constructed as $(\{1, \dots, n\}, \cdot)$ where the universe is a properly sized set of positive integers and

$$x \cdot y = \begin{cases} 1 & \text{for } x = y, \\ \max_{\mathbb{Z}}(x, y) & \text{otherwise.} \end{cases}$$

Using this approach the Fregean order will be the inverse of the usual ordering of integers. One can similarly define an algebra on the universe $\{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}\}$ and use minimum in place of maximum when defining the operation.

Using the same approach, we can construct chains in other varieties with unique extensions and a unique two-element algebra. If we take a n -element chain Heyting algebra, then its (\leftrightarrow) -reduct is the n -element chain equivalential algebra.

We will be interested in working with reducts of Heyting algebras that are locally finite and Fregean. The classic approach to obtain such a structure is by removing the \vee operation.

Definition 1.80. Let $\mathcal{BS}, \mathcal{BS}_0$ denote the classes of all $(\wedge, \rightarrow, 1)$ - and $(\wedge, \rightarrow, 1, 0)$ -subreducts of \mathcal{H} , respectively. The algebras in \mathcal{BS} are called *Brouwerian semilattices*, we call \mathcal{BS}_0 *Brouwerian semilattices with zero* (in the literature they appear also as implicative semilattices and bounded implicative semilattices).

These two classes were thoroughly investigated in the twentieth century. More information about them can be found, for example, in [16, 21].

Lemma 1.81. $\mathcal{BS}, \mathcal{BS}_0$ are congruence permutable Fregean varieties. For any \mathbf{A} in either of those classes, $\text{Univ}(\mathbf{A}) \setminus \{*\}$ is a subuniverse. All nontrivial algebras in both classes have unique extensions.

Proof. Congruence permutability and 1-regularity is preserved by taking subreducts (congruences of a reduct form a sublattice of congruences of original algebra). Principal congruences in \mathcal{BS} and \mathcal{BS}_0 can be defined using the same conditions as in Heyting algebras:

$$\Theta(1, a) = \{(x, y) : a \leq (x \leftrightarrow y)\}.$$

If $(1, b) \in \Theta(1, a)$, then $b \wedge a = (1 \rightarrow b) \wedge a = a$, so if $\Theta(1, b) = \Theta(1, a)$ then $a = a \wedge b = b$ yielding congruence orderability.

To verify that removing $*$ gives a subalgebra, one can either use Theorem 1.65 directly or check from the lattice definition of Heyting algebras that $x \rightarrow y = *$ is only possible for $x = 1, y = *$ and obviously $x \wedge y = *$ implies that one of the variables is $*$.

For any $\mathbf{B} \in \mathcal{BS}$ or \mathcal{BS}_0 we would like to construct \mathbf{B}^\oplus . By Lemma 1.68 such a larger algebra must be constructed by adding a new element $*$ and extending operations in such a way that $(1, *)$ generates the monolith. For any $x \neq 1$ if $x \geq *$, then $\Theta(1, *)$ would have an equivalence class with at least three elements $1, *, x$; if $x, *$ are incomparable, then $x, x \wedge *$ are two different elements that are in the same equivalence class with respect to $\Theta(1, *)$. In both situations $\Theta(1, *)$ cannot be the monolith, so the only way is to add $*$ as the second-largest element, hence the extension is unique up to isomorphism. \square

Remark 1.82. $\mathcal{BS}, \mathcal{BS}_0$ are congruence distributive with $t(x, y, z) = ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow z) \rightarrow z) \wedge ((z \rightarrow x) \rightarrow x)$ being the majority term. As both classes are congruence permutable and congruence distributive, all their finite members are of type **3**.

Later we will find it useful to understand the structure of free \mathcal{BS}_0 over one and two generators. A detailed construction in the general case can be found in [7] as a modification of the construction for \mathcal{BS} . Here we will omit some details, as we are not interested in a higher number of generators, and we will use a slightly modernized approach.

For any algebra \mathbf{A} we can define

$$M' : \text{Con}(\mathbf{A}) \ni \alpha \mapsto M'(\alpha) = \{\theta \in \text{Cm}(\mathbf{A}) : \alpha \leq \theta\}$$

and, as the lattice of congruences is algebraic, $\alpha = \bigwedge M'(\alpha)$ which implies injectivity. If we are operating on congruence orderable algebras (with respect to a common term 1), we can use a similar function but defined on elements instead of congruences

$$M : \text{Univ } \mathbf{A} \ni a \mapsto M(a) = M'(\Theta_{\mathbf{A}}(1, a)).$$

This way we map elements of an algebra to upward-closed subsets of completely meet-irreducible congruences. Moreover, as shown in [25], if considered algebra is from a congruence distributive Fregean variety, then

$$M'(\Theta_{\mathbf{A}}(a, b)) = ((M(a) \div M(b)) \downarrow)'$$

Moreover, if the algebra is finite, then M is actually onto $\text{Up}(\text{Cm}(\mathbf{A}))$. This lets us recover the principal congruence term in $\text{Up}(\text{Cm}(\mathbf{A}))$ using the formula $a \leftrightarrow b = ((a \div b) \downarrow)'$, and M becomes a equivalential algebra isomorphism.

As we are interested in \mathcal{BS}_0 , which is Fregean and congruence distributive, we can use the above to construct free algebras from the poset of their meet-irreducible congruences. If we take \mathbf{F}_1 to be the $\{x\}$ -generated \mathcal{BS}_0 , then for any completely meet-irreducible congruence φ the quotient algebra \mathbf{F}_1/φ is subdirectly irreducible. Due to 1-regularity each such congruence can be uniquely identified by its quotient algebra together with the information which generators were mapped to which elements in the quotient. Brouwerian semilattices with zero satisfy the condition (\dagger) so if \mathbf{F}_1/φ has more than two elements, then it has the second-largest element $*$. In that case, because $(\mathbf{F}_1/\varphi) \setminus \{*\}$ is a subuniverse we must have $x/\varphi = *$. Therefore, $\mathbf{F}_1/\varphi \setminus \{*\}$ is generated by an empty set of generators, hence \mathbf{F}_1/φ is a three-element chain. On the other hand, if \mathbf{F}_1/φ is two-element, then its universe is $\{1, 0\}$ and the image of x can be any of the two. The congruences are ordered by inclusion, so $\varphi \leq \psi$ iff \mathbf{F}_1/φ has a congruence α such that $(\mathbf{F}_1/\varphi)/\alpha \simeq \mathbf{F}_1/\psi$ and $(x/\varphi)/\alpha = x/\psi$. We can conclude that subdirectly irreducible quotient of \mathbf{F}_1 has one of three possible forms. Because of the universal property of free algebras, each of those three forms must be actually realized by one of $\text{Cm}(\mathbf{F}_1)$. As only two of these congruences are comparable, it follows that there are exactly six elements of \mathbf{F}_1 as depicted on Figure 1.2. Each of those elements correspond to an upward closed subset of $\text{Cm}(\mathbf{F}_1)$.

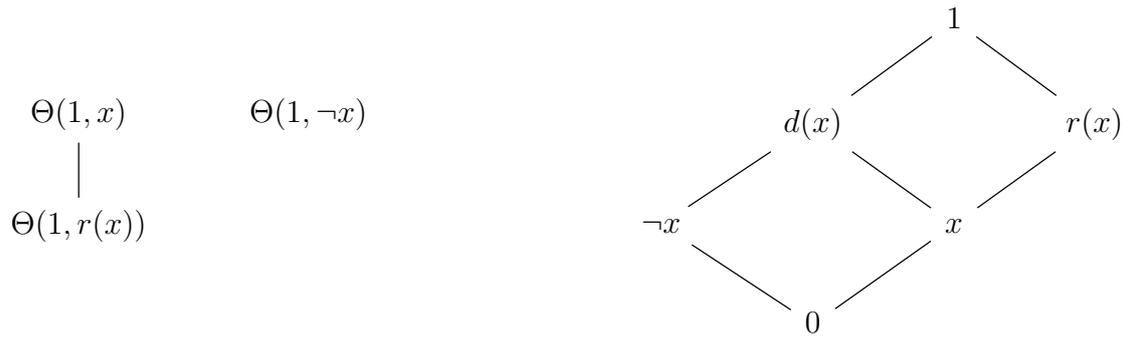


Figure 1.2: The free Brouwerian semilattice with zero over one generator \mathbf{F}_1 . Left: the poset of completely meet-irreducible congruences. Right: the resulting algebra with natural order.

Now, we would like to construct \mathbf{F}_2 , the $\{x, y\}$ -generated \mathcal{BS}_0 . If we take any $\varphi \in \text{Cm}(\mathbf{F}_2)$, then \mathbf{F}_2/φ either has the second-largest element $*$ and one of the generators is mapped to it; or it is two-element. In the first case $(\mathbf{F}_2/\varphi) \setminus \{*\}$ is a subalgebra generated by at most one element (the one not mapped to $*$) so it must be a homomorphic image of \mathbf{F}_1 . In the second case, the quotient algebra has the universe $\{0, 1\}$. As $*$ must be the second-largest element in \mathbf{F}_2/φ , we can recover all 15 possibilities. They are depicted in Figure 1.3.

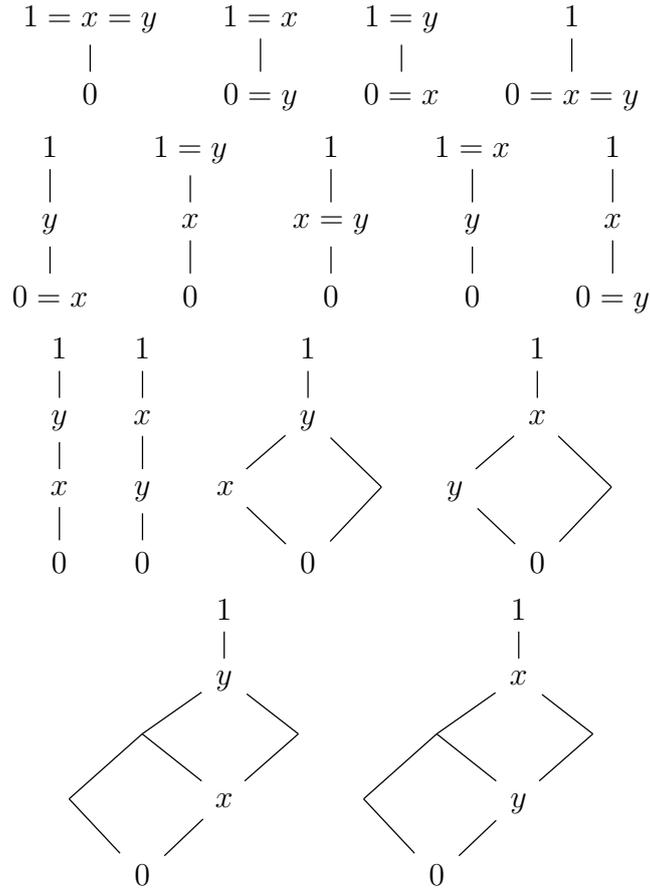


Figure 1.3: Subdirectly irreducible quotients of \mathbf{F}_2 , for brevity x, y denote the cosets containing respective generators.

Define $\mathbf{T}_7 = \mathbf{F}_1^\oplus$, this “test” algebra \mathbf{T}_7 is a useful tool for investigating \mathbf{F}_2 .

Lemma 1.83. We have $\mathbf{F}_2 \in V(\mathbf{T}_7)$ and $\mathbf{T}_7 \in H(\mathbf{F}_2)$. For any two binary terms t, u in the language $(\wedge, \rightarrow, 1, 0)$ we have

$$\mathcal{BS}_0 \models t \approx u \Leftrightarrow \mathbf{T}_7 \models t \approx u.$$

Proof. Every subdirectly irreducible image of \mathbf{F}_2 under homomorphism is isomorphic to a subalgebra of \mathbf{T}_7 . As \mathbf{F}_2 is isomorphic to a subdirect product of its subdirectly irreducible quotients, it is also a subdirect product of subalgebras of \mathbf{T}_7 . This implies $\mathbf{F}_2 \in V(\mathbf{T}_7)$, the other inclusion is true because \mathbf{T}_7 is generated by two elements. \square

By ordering the quotients of \mathbf{F}_2 accordingly, we recover the poset $\text{Cm}(\mathbf{F}_2)$ as depicted in Figure 1.4. Again, each of those algebras must actually be an image of \mathbf{F}_2 because it has the universal property over all two-generated algebras.

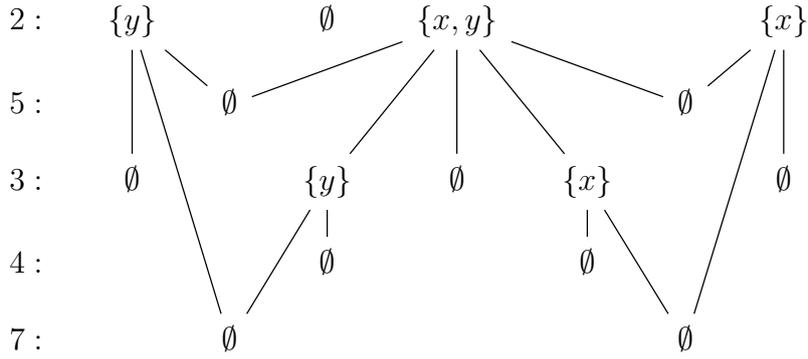


Figure 1.4: The poset $\text{Cm}(\mathbf{F}_2)$, label of a congruence θ indicates which generators are in $1/\theta$. Row labels denote sizes of \mathbf{F}_2/θ .

Theorem 1.84. *The (\leftrightarrow) -reduct of \mathbf{F}_2 is isomorphic to $(\text{Up}(\text{Cm}(\mathbf{F}_2)), \leftrightarrow)$, where $\text{Cm}(\mathbf{F}_2)$ is a known 15-element poset depicted in Figure 1.4 and $a \leftrightarrow b = ((a \div b) \downarrow)'$.*

If we count upward closed subsets of $\text{Cm}(\mathbf{F}_2)$, we obtain, that there are 2134 elements in \mathbf{F}_2 . This is the same value as obtained in [7] and it can also be verified computationally. Thanks to Lemma 1.83, every binary term in \mathcal{BS}_0 is a binary function on \mathbf{T}_7 , which is a composition of fundamental operations and projections. The computational approach gives some insight into \mathbf{F}_2 and we could perform computer-assisted proofs. However, we decided against it and only rely on computer results as a source of conjectures that we later verified.

1.6 Equivalential algebras with regularization

In a Heyting algebra, the elements that are invariant under double negation are called regular. Those elements have multiple additional properties. In particular, they form a subalgebra that is a Boolean algebra (with a different \vee than in the whole algebra). The structure of regular elements will be useful when investigating subreducts of Heyting algebras, which is why we want to introduce the class of algebras investigated in [22].

Definition 1.85 ([22, Definition 3.1]). Algebra (A, \cdot, r) with one binary and one unary operation is called *equivential algebra with regularization* if (A, \cdot) is an equivential algebra and the following identities are satisfied

R1. $r(r(x)) \approx r(x)$;

R2. $r(xy) \approx r(x)r(y)$;

R3. $r(x)yy \approx r(x)$;

R4. $xr(x)r(x) \approx x$.

Theorem 1.86 ([22, Theorem 4.1]). *The class \mathcal{E}_r of all equivalential algebras with regularization is a congruence permutable Fregean variety.*

Example 1.87. Any equivalential algebra (A, \cdot) extended by adding $r(x) = 1$ is an equivalential algebra with regularization.

Any $(\leftrightarrow, r_{\mathcal{H}})$ -reduct of a Heyting algebra is an equivalential algebra with regularization.

As a matter of fact, the second case includes nearly all such algebras. As shown in [22, Theorem 5.1] \mathcal{E}_r is actually the class of all $(\leftrightarrow, r_{\mathcal{H}})$ -subreducts of Heyting algebras.

Corollary 1.88 ([22, Proposition 4.4]). *There are two simple algebras in \mathcal{E}_r : $\mathbf{2}_r = (\{1, *\}, \cdot, x \mapsto x)$ and $\mathbf{2}_d = (\{1, *\}, \cdot, x \mapsto 1)$. If $\mathbf{A} \in \mathcal{E}_r$ is subdirectly irreducible, then we either have $\mathbf{A} \models r(*) = 1$ or $\mathbf{A} \simeq \mathbf{2}_r$.*

Corollary 1.89. *If $\mathbf{A} \in \mathcal{E}_r$ is subdirectly irreducible and there exists x such that $r(x) = *$, then $\mathbf{A} \simeq \mathbf{2}_r$.*

Proof. If $r(x) = *$, then $r(*) = r(r(x)) \stackrel{R1}{=} r(x) = *$. By the previous corollary if $r(*) \neq 1$, then $\mathbf{A} \simeq \mathbf{2}_r$. □

Lemma 1.90 ([22, Proposition 4.6]). Every algebra $\mathbf{A} \in \mathcal{E}_r$ with at least two elements has a unique extension.

From the proof of [22, Theorem 4.1], one can also extract the following property.

Lemma 1.91. Let $\mathbf{A} \in \mathcal{E}_r$, $a \in \mathbf{A}$, $r(a) = a$. The principal congruence generated by $(1, a)$ is

$$\Theta_{\mathbf{A}}(1, a) = \{(x, y) : xyaa \in \{1, a\}\}.$$

Now we would like to present some additional identities of \mathcal{E}_r . We use this lemma as an opportunity to show a particular reasoning method useful in locally finite Fregean varieties, the proof by “lack of a counterexample”. We decided to avoid this method later in the dissertation if possible, because sometimes it is hard to make it into a readable text. Moreover, this approach cannot be used until we are sure we work in a locally finite Fregean variety, so in particular we cannot use it while proving Fregeanity itself.

Lemma 1.92. The following identities are valid in all equivalential algebras with regularization:

$$\text{R5. } xr(y)r(z) \approx xr(z)r(y);$$

$$\text{R6. } xr(x)r(y) \approx xr(xy);$$

$$\text{R7. } xy \approx (xr(x))(yr(y))r(xy).$$

Proof. By the axioms of \mathcal{E}_r we have $r(xyy) \stackrel{R2}{=} r(x)r(y)r(y) \stackrel{R3}{=} r(x) \stackrel{R3}{=} r(x)yy$, so we can apply 1.65 to show that this is a locally finite variety.

Assume that an identity is false. There exists an algebra \mathbf{A} and elements $x, y, z \in \mathbf{A}$ that form a counterexample. Subalgebra of \mathbf{A} that is generated by x, y, z is also a counterexample; additionally it is also finite. There exists at least one finite counterexample, so it is well defined to take a minimal counterexample with respect to the size of universe. We can assume without loss of generality that \mathbf{A} is minimal. A minimal example must be subdirectly irreducible, or one of its quotients would be a smaller counterexample. On the other hand, minimality implies that $\mathbf{A}/\mu_{\mathbf{A}}$ is not a counterexample. But the only nontrivial class of $\mu_{\mathbf{A}}$ is $\{1, *\}$. Hence, the elements x, y, z for which the identity fails must be such that after substitution one side of equality is equal to 1 and the other to *. We can now use this observation to reverse engineer the structure of \mathbf{A} and identify elements for which the identity does not hold.

Notice that all three identities are satisfied in $\mathbf{2}_r$ because in that algebra $r(x) \approx x$. By Corollary 1.88 we have $\mathbf{A} \models r(*) = 1$, so by Corollary 1.67, the only solution to $r(x)y = *$ is $r(x) = 1, y = *$.

Assume that R5 is false. There exist elements $x, y, z \in \mathbf{A}$ such that $xr(y)r(z) = *, xr(z)r(y) = 1$ (this is without loss of generality, as we can swap y and z). First, equality is possible only if $x = *, r(y) = r(z) = 1$, but then the right-hand side is also equal to $*$. Similarly, if R6 is false for some elements, then the only possibility is $x = *$. Then we would have $xr(x)r(y) = *r(*)r(y) = *1r(y) = *r(y) = *(1r(y)) = *(r(*)r(y)) \stackrel{R2}{=} *r(*) = xr(xy)$ so R6 cannot be false. If we assume that R7 is false in \mathcal{A} , then either $xy = 1$ or $xy = *$. In the first case $x = y$, so $(xr(x))(yr(y))r(xy) = (xr(x))(xr(x))r(xx) = r(1) = 1 = xy$. In the second case without loss of generality $x = 1, y = *$ and $(xr(x))(yr(y))r(xy) = (1r(1))(r(*)r(1*)) = *r(*)r(*) = *11 = * = xy$. Hence, all three identities hold in finite algebras; they must also hold in the whole variety. \square

The unary operator r defines two subuniverses in a natural way.

Lemma 1.93. Let $(A, \cdot, r) \in \mathcal{E}_r$ and $d(x) = x \cdot r(x)$. The sets $r(A), d(A)$ are subuniverses. $(r(A), \cdot)$ is a Boolean group (\cdot is associative). The set $d(A) = \{d(x) : x \in A\}$ is equal to $\{x \in A : x = d(x)\}$ and to $\{x \in A : r(x) = 1\}$.

Proof. The set $r(A)$ is a subuniverse by R1 and R2. The \cdot on this set is associative by R5. We have the following implications

$$\begin{aligned} y = d(x) &\Rightarrow r(y) = r(xr(x)) \stackrel{R2}{=} r(x)r(r(x)) \stackrel{R1}{=} r(x)r(x) = 1, \\ r(y) = 1 &\Rightarrow y \stackrel{R4}{=} yr(y)r(y) = d(y)r(y) = d(y), \\ y = d(y) &\Rightarrow \exists_x y = d(x), \end{aligned}$$

which shows that $d(A) = \{x \in A : x = d(x)\} = \{x \in A : r(x) = 1\}$. By R2, the set $\{x \in A : r(x) = 1\}$ is closed under \cdot . We also see that $r(d(x)) = 1 = d(1) \in d(A)$, showing that $d(A)$ is a subuniverse. \square

Chapter 2

Equivalential algebras with partial semilattice

This chapter presents two classes of subreducts of Heyting algebras that contain algebras of a mixed type. Understanding those two was an inspiration for all other research in this thesis.

From the Corollary 1.72 we have that on a finite universe there are only finitely many ways (up to polynomial equivalence) an algebra can be defined, such that it generates a congruence permutable Fregean variety. In a two-element universe, there are two possible clones of polynomials, differing in the value of the commutator $[1_{\mathbf{A}}, 1_{\mathbf{A}}] \in \{1_{\mathbf{A}}, 0_{\mathbf{A}}\}$. These two clones can be obtained, for example, from a two-element equivalential algebra ($[1_{\mathbf{A}}, 1_{\mathbf{A}}] = 0_{\mathbf{A}}$) and a two-element Brouwerian semilattice ($[1_{\mathbf{A}}, 1_{\mathbf{A}}] = 1_{\mathbf{A}}$). Neither equivalential algebras nor Brouwerian semilattices are of mixed type, so they are not in the scope of our research. The case of a three-element universe was considered by Sławomir Przybyło in his thesis, where he came to the conclusion that there are four possible clones of polynomials. Algebras that have such clones of polynomials can be obtained as reducts of the three-element Heyting algebra. These are the following

- (\leftrightarrow) -reduct (equivalential algebra);
- (\rightarrow, \wedge) -reduct (Brouwerian semilattice);
- $(\leftrightarrow, (x, y) \mapsto r_{\mathcal{H}}(x) \wedge r_{\mathcal{H}}(y))$ -reduct (denoted \mathbf{R});

- $(\leftrightarrow, (x, y) \mapsto d_{\mathcal{H}}(x) \wedge d_{\mathcal{H}}(y))$ -reduct (denoted \mathbf{D}).

Two chapters of the mentioned dissertation were devoted to the study of varieties $\mathcal{V}(\mathbf{R}), \mathcal{V}(\mathbf{D})$. The current chapter is devoted to the study of the two varieties generated by analogous reducts of Heyting algebras, but without limiting ourselves to the three-element universe. Of course, $\mathcal{V}(\mathbf{R}), \mathcal{V}(\mathbf{D})$ will be subvarieties of them. We would like to point out that both \mathbf{D} and \mathbf{R} are of mixed type.

2.1 Semilattice of regular elements

The results contained in this section are already published as a joint work with my advisor, Katarzyna Korwin-Słomczyńska [18].

In a Heyting algebra we can define the binary operation $x \widehat{\wedge} y = r_{\mathcal{H}}(x) \wedge r_{\mathcal{H}}(y)$, on regular elements this operation agrees with the ordinary meet. It also allows us to define the unary regularization operation $r_{\mathcal{H}}(x) = x \widehat{\wedge} x$. If we consider a three-element Heyting algebra (a three-element chain with natural operations), its $(\leftrightarrow, \widehat{\wedge})$ -reduct is the algebra \mathbf{R} investigated by Przybyło. Our goal in this section is to show that the $(\leftrightarrow, \widehat{\wedge})$ -subducts of Heyting algebras constitute a variety. We will indicate a set of identities that are satisfied by these subreducts and then show that any algebra which satisfies these identities is indeed a subreduct of a Heyting algebra.

Definition 2.1. An *equivalential algebra with semilattice of regular elements* (or *EARS* for short) is an algebra $(A, \cdot, \widehat{\wedge})$ with two binary operations satisfying the following identities (for the sake of brevity, we use $r(x) = x \widehat{\wedge} x$):

E1. $xy \approx y$;

E2. $xyz \approx (xz)(yz)$;

E3. $xy(xzz)(xzz) \approx xy$;

S1. $r(x) \widehat{\wedge} y \approx x \widehat{\wedge} y$;

S2. $x \widehat{\wedge} y \approx y \widehat{\wedge} x$;

$$\text{S3. } (x \hat{\wedge} y) \hat{\wedge} z \approx x \hat{\wedge} (y \hat{\wedge} z);$$

$$\text{M1. } r(xx) \approx xx;$$

$$\text{M2. } xr(x)r(x) \approx x;$$

$$\text{M3. } r(x)yy \approx r(x);$$

$$\text{M4. } (x \hat{\wedge} z)(y \hat{\wedge} z)r(z) \approx (xy) \hat{\wedge} z;$$

$$\text{M5. } x(y \hat{\wedge} z)(y \hat{\wedge} z) \approx xr(z)r(z)r(y)r(y).$$

We denote by \mathcal{V}_{EARS} the variety of such algebras.

The axioms E1-E3 make (A, \cdot) an equivalential algebra. S1-S3 introduce a semilattice structure on $(r(A), \hat{\wedge})$. M1-M5 describe how these two structures are mixed together. M1-M3 are known properties of regular elements in Heyting algebras. In a moment, we would show that M1-M5 in particular imply that (A, \cdot, r) is an equivalential algebra with regularization. Our goal is to show that \mathcal{V}_{EARS} is the class of all appropriate subreducts of Heyting algebras, first we will show the easier inclusion.

Lemma 2.2. $(\leftrightarrow, \hat{\wedge})$ -reducts of Heyting algebras are EARS.

Proof. The only thing we need to show is that the listed identities hold in Heyting algebras. Identities E1-E3 hold because (\leftrightarrow) -reducts of Heyting algebras are equivalential algebras. S1-S3 describe the semilattice structure of regular elements, so they follow from \wedge being a semilattice operation and the set of regular elements being closed under \wedge . If we remember that in Heyting algebras we can define regularization by $r_{\mathcal{H}}(x) = x00$ then $r_{\mathcal{H}}(xx) = xx00 \stackrel{E1}{=} 00 = 1 = xx$ and $xr_{\mathcal{H}}(x)r_{\mathcal{H}}(x) = x(x00)(x00) \stackrel{E3}{=} x$. We also have $r_{\mathcal{H}}(xy) = xy00 \stackrel{E10}{=} (x00)(y00) = r_{\mathcal{H}}(x)r_{\mathcal{H}}(y)$, so M4 can be rewritten to a statement that

$$(x \wedge z)(y \wedge z)z = (xy) \wedge z$$

for all regular x, y, z . This equality is a tautology of classical logic, so it is true for regular elements (regular elements form a Boolean algebra). M3 follows from M5 with $z = 0$.

To show M5, we will show that the more general equality

$$x(y \wedge z)(y \wedge z) = xy yzz$$

is true in Heyting algebras. Assuming this equality is false in some Heyting algebra, it must also be false in its $(\wedge, \rightarrow, 1)$ -reduct, which is a Brouwerian semilattice. This means that it is not an identity of \mathcal{BS} . We will now perform a proof by the “lack of a counterexample”. Using the reasoning described in Lemma 1.92, if the identity is false, then there exists a subdirectly irreducible counterexample algebra \mathbf{A} and elements $x, y, z \in \mathbf{A}$ such that one side of the equality is 1 and the other is $*$.

Due to Corollary 1.67 $x(y \wedge z)(y \wedge z) = *$ is only possible with $x = *, y \wedge z \in \{1, *\}$. This implies $y, z \in \{1, *\}$, so the right-hand side of the equality is one of the following possibilities: $*1111, *11 ** , ** *11, * * * * *$. However, each of those is equal to $*$, so it is equal to the left-hand side. Similarly, if we assume $xy yzz = *$ then $x = *, y, z \in \{1, *\}$ so $y \wedge z \in \{1, *\}$ and the left-hand side is also equal to $*$. This shows that the equality holds in any finite \mathcal{BS} , so it holds in the whole variety, and it must also hold in \mathcal{H} . \square

Now, we would like to explain why M4 and M5 are necessary. To show that they do not follow from the other identities, we will construct examples of algebras that satisfy all other identities but not one of them. This means that removing either of them from the definition would yield a larger class. In particular, this class would contain at least one algebra that is not a subreduct of a Heyting algebra.

Example 2.3. Take the set $A = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$. Define $\widehat{\wedge}$ as the minimum operation in natural order and \cdot as a Boolean group operation with neutral element 1. The algebra $\mathbf{A} = (A, \cdot, \widehat{\wedge})$ violates M4 because

$$(0 \widehat{\wedge} \frac{1}{3}) (\frac{2}{3} \widehat{\wedge} \frac{1}{3}) (\frac{1}{3} \widehat{\wedge} \frac{1}{3}) = 0 \cdot \frac{1}{3} \cdot \frac{1}{3} = 0 \neq \frac{1}{3} = \frac{1}{3} \widehat{\wedge} \frac{1}{3} = (0 \cdot \frac{2}{3}) \widehat{\wedge} \frac{1}{3}.$$

However in \mathbf{A} we have $r(x) \approx x \approx xyy$, which lets us easily verify that all other axioms hold in \mathbf{A} . Because \mathbf{A} violates M4, it cannot be a $(\leftrightarrow, \widehat{\wedge})$ -subreduct of a Heyting algebra.

Example 2.4. Consider a Heyting algebra which is a product of two chains $\{1, 0\} \times \{1, *, 0\}$. We take its equivalential reduct (A, \cdot) and define $\widehat{\wedge}$ to be the meet-semilattice operation on regular elements such that $(1, 1)$ is the largest, $(0, 1)$ the smallest and $(1, 0), (0, 0)$ are incomparable. Finally, we extend $\widehat{\wedge}$ to $A \times A$ in a way that $r((x, *)) = (x, 1)$ and S1-S3 hold. In this way, we obtain an algebra $(A, \cdot, \widehat{\wedge})$ that satisfies all identities in the definition of EARS with the exception of M5. The M5 does not hold because

$$\begin{aligned}
(1, *)((0, 0)\widehat{\wedge}(1, 0))((0, 0)\widehat{\wedge}(1, 0)) &= (1, *) (0, 1)(0, 1) \\
&= (1, *) \neq (1, 1) \\
&= (1, 1)(1, 0)(1, 0) \\
&= (1, *) (0, 0)(0, 0)(1, 0)(1, 0) \\
&= (1, *)r((0, 0))r((0, 0))r((1, 0))r((1, 0)).
\end{aligned}$$

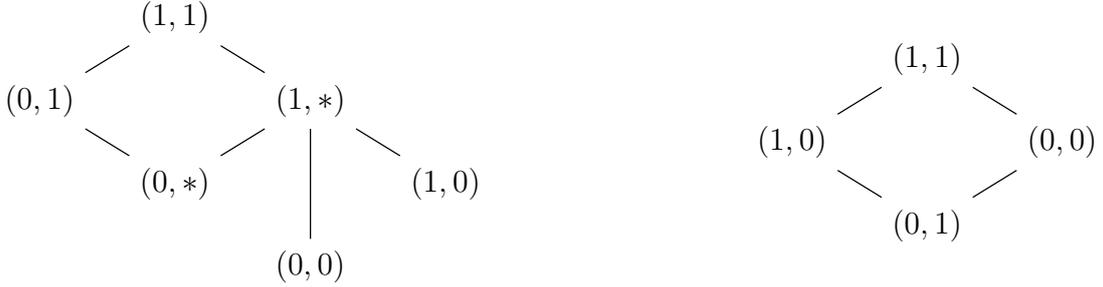


Figure 2.1: The algebra from Example 2.4. Left: the natural order on (A, \cdot) . Right: order induced by $\widehat{\wedge}$.

Now we will show some additional identities that follow from the definition.

Lemma 2.5. The following identities are also true in \mathcal{V}_{EARS} .

U1. $x\widehat{\wedge}1 \approx r(x)$;

U2. $r(x\widehat{\wedge}y) \approx x\widehat{\wedge}y$;

U3. $r(x)r(y) \approx r(xy)$;

U4. $r(r(x)) \approx r(x)$.

Proof. To show that each identity holds, we perform a series of substitutions. For any $\mathbf{A} \in \mathcal{V}_{EARS}$ and elements $x, y, z \in \mathbf{A}$ we have $x \widehat{\wedge} (yy) \stackrel{M4}{=} (y \widehat{\wedge} x)(y \widehat{\wedge} x)r(x) \stackrel{E1}{=} r(x)$. Similarly we can show every other identity in order:

$$\begin{aligned} r(x \widehat{\wedge} y) &= (x \widehat{\wedge} y) \widehat{\wedge} (x \widehat{\wedge} y) \stackrel{S3, S2}{=} (x \widehat{\wedge} x) \widehat{\wedge} (y \widehat{\wedge} y) = r(x) \widehat{\wedge} r(y) \stackrel{S1, S2}{=} x \widehat{\wedge} y, \\ r(x)r(y) &\stackrel{E1}{=} r(x)r(y)1 \stackrel{U1}{=} (x \widehat{\wedge} 1)(y \widehat{\wedge} 1)r(1) \stackrel{M4}{=} (xy) \widehat{\wedge} 1 \stackrel{U1}{=} r(xy), \\ r(r(x)) &= r(x) \widehat{\wedge} r(x) \stackrel{S1}{=} x \widehat{\wedge} r(x) \stackrel{S2}{=} r(x) \widehat{\wedge} x \stackrel{S1}{=} x \widehat{\wedge} x = r(x). \end{aligned}$$

□

Remark 2.6. The above lemma implies that the (\cdot, r) -reduct of an EARS is an equivalential algebra with regularization (Definition 1.85). Therefore, by Lemma 1.92, the identities R5-R7 are also true in \mathcal{V}_{EARS} .

Our main goal is to show the following theorem.

Theorem 2.7. *The variety \mathcal{V}_{EARS} is equal to the class \mathcal{H}_{emd} of all $(\leftrightarrow, \widehat{\wedge})$ -subreducts of Heyting algebras.*

All identities defining EARS are true in Heyting algebras, so $\mathcal{H}_{emd} \subset \mathcal{V}_{EARS}$. To prove the other inclusion, we would like to use some results regarding congruence permutable Fregean varieties. But first, we need to show that \mathcal{V}_{EARS} is such a variety.

Lemma 2.8. For any $\mathbf{A} = (A, \cdot, \widehat{\wedge}) \in \mathcal{V}_{EARS}$ and $a, b \in A, r(b) = 1$ the following holds:

- (a) $\Theta_{\mathbf{A}}(1, a) \subset \{(x, y) : x \widehat{\wedge} a = y \widehat{\wedge} a\} \in \text{Con}(\mathbf{A})$;
- (b) $\Theta_{\mathbf{A}}(1, r(a)) = \{(x, y) : x \widehat{\wedge} a = y \widehat{\wedge} a, xr(x)r(a)r(a) = yr(y)r(a)r(a)\}$;
- (c) $\Theta_{\mathbf{A}}(1, a) = \Theta_{\mathbf{A}}(1, ar(a)) \vee \Theta_{\mathbf{A}}(1, r(a))$;
- (d) $\Theta_{\mathbf{A}}(1, b) = \Theta_{\mathbf{A}^e}(1, b)$ where \mathbf{A}^e denotes the (\cdot) -reduct of \mathbf{A} .

Proof. (a): Consider the set $T_a = \{(x, y) : x \widehat{\wedge} a = y \widehat{\wedge} a\}$, which of course is an equivalence relation. This set preserves $\widehat{\wedge}$, because $x \widehat{\wedge} a = y \widehat{\wedge} a$ implies

$$(z \widehat{\wedge} x) \widehat{\wedge} a \stackrel{S3}{=} z \widehat{\wedge} (x \widehat{\wedge} a) = z \widehat{\wedge} (y \widehat{\wedge} a) \stackrel{S3}{=} (z \widehat{\wedge} y) \widehat{\wedge} a.$$

It also preserves \cdot , because $x\hat{\wedge}a = y\hat{\wedge}a$ implies

$$(zx)\hat{\wedge}a \stackrel{M4}{=} (x\hat{\wedge}a)(z\hat{\wedge}a)r(a) = (y\hat{\wedge}a)(z\hat{\wedge}a)r(a) \stackrel{M4}{=} (zy)\hat{\wedge}a.$$

This means that $T_a \in \text{Con}(\mathbf{A})$. We also have $(1, a) \in T_a$, so $\Theta_{\mathbf{A}}(1, a) \subset T_a$.

(b): Fix $a \in \mathbf{A}$ and take

$$\theta_a = \{(x, y) : x\hat{\wedge}a = y\hat{\wedge}a, xr(x)r(a)r(a) = yr(y)r(a)r(a)\}.$$

We want to show that it is a congruence, from its definition it is obvious that it is an equivalence relation. Take $(x, y) \in \theta_a, z \in A$, looking at the previous point we have $(zx)\hat{\wedge}a = (zy)\hat{\wedge}a$ and $(z\hat{\wedge}x)\hat{\wedge}a = (z\hat{\wedge}y)\hat{\wedge}a$. Now we compute

$$\begin{aligned} (x\hat{\wedge}z)r(x\hat{\wedge}z)r(a)r(a) &\stackrel{U2}{=} r(a)r(a) \stackrel{U2}{=} (y\hat{\wedge}z)r(y\hat{\wedge}z)r(a)r(a), \\ xzr(xz)r(a)r(a) &\stackrel{R7}{=} xr(x)(zr(z))r(xz)r(xz)r(a)r(a) \\ &\stackrel{E8}{=} xr(x)(zr(z))r(a)r(a)r(xz)r(xz) \\ &\stackrel{E10}{=} xr(x)r(a)r(a)(zr(z)r(a)r(a))r(xz)r(xz) \\ &= yr(y)r(a)r(a)(zr(z)r(a)r(a))r(xz)r(xz) \\ &\stackrel{E10}{=} yr(y)(zr(z))r(a)r(a)r(xz)r(xz) \\ &\stackrel{M5}{=} yr(y)(zr(z))(a\hat{\wedge}(xz))(a\hat{\wedge}(xz)) \\ &= yr(y)(zr(z))(a\hat{\wedge}(yz))(a\hat{\wedge}(yz)) \\ &\stackrel{M5}{=} yr(y)(zr(z))r(yz)r(yz)r(a)r(a) \\ &\stackrel{R7}{=} yzr(yz)r(a)r(a), \end{aligned}$$

showing that θ_a is a congruence. Now take any $(x, y) \in \theta_a$, we have

$$\begin{aligned} r(x) &\stackrel{U1}{=} x\hat{\wedge}1 \equiv_{\Theta_{\mathbf{A}}(1, r(a))} x\hat{\wedge}a \\ &= y\hat{\wedge}a \equiv_{\Theta_{\mathbf{A}}(1, r(a))} y\hat{\wedge}1 \stackrel{U1}{=} r(y), \\ x &\stackrel{M2}{=} xr(x)r(x) \equiv_{\Theta_{\mathbf{A}}(1, r(a))} xr(x)r(a)r(a)r(x) \\ &= yr(y)r(a)r(a)r(y) \equiv_{\Theta_{\mathbf{A}}(1, r(a))} yr(y)r(y) \stackrel{M2}{=} y, \end{aligned}$$

so $(x, y) \in \Theta_{\mathbf{A}}(1, r(a))$, showing that $\Theta_{\mathbf{A}}(1, r(a)) = \theta_a$.

(c): One inclusion stems from the fact that congruences preserve operations

$$\Theta_{\mathbf{A}}(1, a) \supset \Theta_{\mathbf{A}}(1, ar(a)), \Theta_{\mathbf{A}}(1, a) \supset \Theta_{\mathbf{A}}(1, r(a));$$

$$\Theta_{\mathbf{A}}(1, a) \supset \Theta_{\mathbf{A}}(1, ar(a)) \vee \Theta_{\mathbf{A}}(1, r(a)).$$

On the other hand, $(1, a) \stackrel{M2}{=} (1, ar(a)r(a)) \in \Theta_{\mathbf{A}}(1, ar(a)) \vee \Theta_{\mathbf{A}}(1, r(a))$.

(d): A congruence generated by a fixed set can only become smaller if we take a reduct, so $\Theta_{\mathbf{A}^e}(1, b) \subset \Theta_{\mathbf{A}}(1, b)$. As $r(b) = 1$, we have $\Theta_{\mathbf{A}^e}(1, b) \subset \ker r$. Take $x, y, z \in A$ such that $(x, y) \in \Theta_{\mathbf{A}^e}(1, b)$, it implies $(x, y) \in \ker r$ so $r(x) = r(y)$. It follows that $x \widehat{\wedge} z \stackrel{S1}{=} r(x) \widehat{\wedge} z = r(y) \widehat{\wedge} z \stackrel{S1}{=} y \widehat{\wedge} z$ and $(x \widehat{\wedge} z, y \widehat{\wedge} z) = (x \widehat{\wedge} z, x \widehat{\wedge} z) \in \Theta_{\mathbf{A}^e}(1, b)$. This shows that $\Theta_{\mathbf{A}^e}(1, b)$ preserves $\widehat{\wedge}$, so it is a congruence. By definition, $\Theta_{\mathbf{A}}(1, b)$ is supposed to be minimal. Therefore, $\Theta_{\mathbf{A}}(1, b) \subset \Theta_{\mathbf{A}^e}(1, b)$. \square

Theorem 2.9. *The variety \mathcal{V}_{EARS} is congruence permutable and Fregean.*

Proof. Adding operations to the language of algebra can only remove elements from its lattice of congruences, so \mathcal{V}_{EARS} is congruence permutable and 1-regular because \mathcal{E} is. To show congruence orderability fix an algebra $\mathbf{A} = (A, \cdot, \widehat{\wedge}) \in \mathcal{V}_{EARS}$ and two elements $a, b \in A$ such that $\Theta_{\mathbf{A}}(1, a) = \Theta_{\mathbf{A}}(1, b)$. By point (a) of the previous lemma, we have $(1, a) \in \Theta_{\mathbf{A}}(1, b) \subset \{(x, y) : x \widehat{\wedge} b = y \widehat{\wedge} b\}$, so $r(b) \stackrel{U1}{=} 1 \widehat{\wedge} b = a \widehat{\wedge} b$. In a similar manner $r(a) = a \widehat{\wedge} b$, implying $r(a) = r(b)$.

Let $\alpha = \Theta_{\mathbf{A}}(1, r(a))$, by the previous Lemma points (c),(d)

$$\Theta_{\mathbf{A}}(1, a) = \Theta_{\mathbf{A}^e}(1, ar(a)) \vee \alpha = \Theta_{\mathbf{A}^e}(1, br(a)) \vee \alpha = \Theta_{\mathbf{A}}(1, b).$$

In the quotient algebra \mathbf{A}^e/α , from Lemma 1.48 we have

$$\begin{aligned} \Theta_{\mathbf{A}^e/\alpha}(1/\alpha, ar(a)/\alpha) &= (\Theta_{\mathbf{A}^e}(1, ar(a)) \vee \alpha)/\alpha \\ &= (\Theta_{\mathbf{A}^e}(1, br(b)) \vee \alpha)/\alpha \\ &= \Theta_{\mathbf{A}^e/\alpha}(1/\alpha, br(b)/\alpha). \end{aligned}$$

As $\mathbf{A}^e/\alpha \in \mathcal{E}$ it is congruence orderable, so $ar(a)/\alpha = br(b)/\alpha$ and $(ar(a), br(b)) \in \alpha$. By the description of the congruence α from point (b) of the previous lemma, we

obtain $ar(a)r(ar(a))r(a)r(a) = br(b)r(br(b))r(a)r(a)$. However

$$\begin{aligned} ar(a)r(ar(a))r(a)r(a) &\stackrel{U3}{=} ar(a)(r(a)r(r(a)))r(a)r(a) \\ &\stackrel{U4}{=} ar(a)(r(a)r(a))r(a)r(a) \\ &\stackrel{E1}{=} ar(a)r(a)r(a) \\ &\stackrel{E6}{=} ar(a) \end{aligned}$$

and similarly $br(b)r(br(b))r(a)r(a) = br(b)$, hence

$$a \stackrel{M2}{=} ar(a)r(a) = cr(a) = cr(b) = br(b)r(b) \stackrel{M2}{=} b.$$

□

By directly applying Theorem 1.65, we have the following.

Corollary 2.10. \mathcal{V}_{EARS} is locally finite and primitive.

Knowing the above, we can focus on finite algebras in \mathcal{V}_{EARS} . We will start from those with the most basic structure.

Lemma 2.11. \mathcal{V}_{EARS} has exactly two simple algebras (up to isomorphism). Both are subreducts of Heyting algebras.

Proof. By Corollary 1.64, a simple algebra $(A, \cdot, \hat{\wedge}) \in \mathcal{V}_{EARS}$ has two elements, so we can write $A = \{1, *\}$. The operation \cdot can be defined in only one way (as $x = y \Leftrightarrow xy = 1$); for $\hat{\wedge}$ we have

$$1\hat{\wedge}1 = r(1) \stackrel{M1}{=} 1; \quad 1\hat{\wedge}* \stackrel{S2}{=} *\hat{\wedge}1 \stackrel{U1}{=} r(*) = *\hat{\wedge}*.$$

If we assume $r(*) = *$, then we obtain $\mathbf{2}_r$, which is a reduct of the two-element Heyting algebra. Alternatively, if $r(*) = 1$, then we arrive at the algebra $\mathbf{2}_d$ which is a subreduct of the Heyting algebra chain $\{1, *, 0\}$. □

Remark 2.12. In the above proof, we distinguished a specific algebra by the notation $\mathbf{2}_r \in \mathcal{V}_{EARS}$, which seems to conflict with the $\mathbf{2}_r \in \mathcal{E}_r$ we used earlier. This is an intentional abuse of notation. In both cases we use this symbol to denote a two-element algebra that satisfies $r(x) = x$. Notice that $\mathbf{2}_r \in \mathcal{E}_r$ is the (\cdot, r) -reduct

of $\mathbf{2}_r \in \mathcal{V}_{EARS}$ and both are respective reducts of the two-element Heyting algebra. Similarly, we also abuse $\mathbf{2}_d$ to denote a two-element algebra in which $r(x) = 1$.

The algebra $\mathbf{2}_r \in \mathcal{V}_{EARS}$ is polynomially equivalent to a two-element Boolean algebra, so it is of type **3**. On the other hand, $\mathbf{2}_d \in \mathcal{V}_{EARS}$ is polynomially equivalent to a two-element equivalential algebra, so it is of type **2**. Any EARS that has both $\mathbf{2}_r$ and $\mathbf{2}_d$ as subalgebras must have a mixed type. The simplest example would be the reduct of a three-element Heyting chain with universe $\{1, *, 0\}$, this is the algebra \mathbf{R} investigated by Przybyło.

Lemma 2.13. Every nontrivial $\mathbf{A} \in \mathcal{V}_{EARS}$ has a unique extension.

Proof. Let $\mathbf{B} \in \mathcal{V}_{EARS}$ be subdirectly irreducible with $\mathbf{B}/\mu_{\mathbf{B}} \simeq \mathbf{A}$. By Lemma 1.68 \mathbf{A} is isomorphic to a subalgebra of \mathbf{B} which arises by removing $*$ from the universe. By Corollary 1.88, $r^{\mathbf{B}}(*) = 1$ as this algebra has at least three elements. Hence, the only way (up to isomorphism) to obtain \mathbf{B} is by adding to \mathbf{A} a new element $*$ and extending operations in such a way that $r(*) = 1$. As r can be extended in only one way, the same is true for $\widehat{\wedge}$ because

$$x \widehat{\wedge} y = r(x) \widehat{\wedge} r(y).$$

The equivalence can be extended in only one way due to the unique extensions property of \mathcal{E} , so \mathbf{B} is unique up to isomorphism.

By direct verification, the algebra obtained in this way is an EARS (just check that all identities still hold when some of the variables are set to $*$). \square

Theorem 2.14. Any finite $\mathbf{A} = (A, \cdot, \widehat{\wedge}) \in \mathcal{V}_{EARS}$ is a $(\cdot, \widehat{\wedge})$ -subreduct of a finite Heyting algebra.

Proof. We will perform a proof by induction with respect to the size of an algebra. Lemma 2.11 addresses the base case where $|A| = 2$. Fix an algebra with $|A| > 2$ and assume that the theorem is true for all algebras with a smaller universe. If \mathbf{A} is not subdirectly irreducible, then $\mathbf{A} \leq \prod_{i \in I} \mathbf{B}_i$ for some family of algebras \mathbf{B}_i , each of which has a strictly smaller universe than \mathbf{A} . By the inductive hypothesis, each \mathbf{B}_i is a subreduct of a Heyting algebra \mathbf{C}_i , so \mathbf{A} is a subreduct of $\prod_{i \in I} \mathbf{C}_i$.

If \mathbf{A} is subdirectly irreducible, then it has the second-largest element $*_A$. The algebra $\mathbf{B} = (A \setminus \{*_A\}, \cdot, \widehat{\wedge}) \leq \mathbf{A}$ is, by the inductive hypothesis, a subreduct of some Heyting algebra \mathbf{C} . By Lemma 1.76 we can extend \mathbf{C} by adding $*_C$ to the universe. The bigger Heyting algebra \mathbf{C}^\oplus is subdirectly irreducible with the monolith $\mu = \{(x, y) : x = y \text{ or } \{x, y\} = \{1, *_C\}\}$. Let \mathbf{D} be the $(\cdot, \widehat{\wedge})$ -reduct of \mathbf{C}^\oplus , we know that $\mathbf{B} \leq \mathbf{D}$, and μ is a congruence on \mathbf{D} . If we add $*_C$ to the universe of \mathbf{B} , the resulting set must be a subuniverse of \mathbf{D} or we will contradict $\mu \in \text{Con}(\mathbf{D})$. Hence, $\mathbf{A}' = ((A \setminus \{*_A\}) \cup \{*_C\}, \cdot, \widehat{\wedge}) \leq \mathbf{D}$. We have

$$\mathbf{A}/\mu_{\mathbf{A}} \simeq \mathbf{A}'/\mu_{\mathbf{A}'},$$

with the isomorphism given by

$$f(x) = \begin{cases} *_C & \text{when } x = *_A; \\ x & \text{otherwise.} \end{cases}.$$

By unique extensions $\mathbf{A} \simeq \mathbf{A}'$, and by definition \mathbf{A}' is a subreduct \mathbf{C}^\oplus . \square

Proof of theorem 2.7. As mentioned before; $(\cdot, \widehat{\wedge})$ -subreducts of Heyting algebras satisfy the axioms of EARS, so $S(\mathcal{H}_{emd}) \subset \mathcal{V}_{EARS}$. A family of subreducts is always a quasivariety, so by the primitivity of \mathcal{V}_{EARS} , $S(\mathcal{H}_{emd})$ is a variety. All finitely generated members of \mathcal{V}_{EARS} are subreducts of Heyting algebras. If two varieties have the same finitely generated members, then they satisfy the same identities, so $S(\mathcal{H}_{emd}) = \mathcal{V}_{EARS}$. \square

The proof of Theorem 2.7 does not directly use the fact that we are working with EARS, only some of its properties. This leads to the following generalization.

Theorem 2.15. *If a variety:*

- *is congruence permutable;*
- *is Fregean;*
- *nontrivial algebras have unique extensions;*
- *is locally finite (in particular if it satisfies the property (\dagger) from Theorem 1.65);*

- *all its simple algebras are subreducts of Heyting algebras,*

then all its finitely generated algebras are subreducts of Heyting algebras.

2.2 Semilattice of dense elements

Similarly to the previous section, we define a binary operation in the language of Heyting algebras, and this time it is the conjunction on dense elements:

$$x \sqcap y = d_{\mathcal{H}}(x) \wedge d_{\mathcal{H}}(y)$$

(we remind, that $d_{\mathcal{H}}$ is the densification in Heyting algebras, $d_{\mathcal{H}}(x) = r_{\mathcal{H}}(x) \leftrightarrow x = (\neg\neg x) \rightarrow x$).

We would like to describe the variety of all $(\leftrightarrow, \sqcap)$ -subreducts. Reasoning is analogous to the case of $(\leftrightarrow, \widehat{\wedge})$ -subreducts. As before, we start by “guessing” a certain equational class, and gradually we will prove that it is the class of subreducts we wanted.

Definition 2.16. Let $\mathbf{A} = (A, \cdot, \sqcap)$ be an algebra with two binary operations. For brevity, we would write $r(x) = (x \sqcap x) \cdot x$, $d(x) = x \sqcap x$. We call \mathbf{A} an *equivalential algebra with semilattice of dense elements* or *EADS*, if (A, \cdot, r) is an equivalential algebra with regularization (so it satisfies identities E1-E3, R1-R4) and moreover the following identities hold:

$$\text{S'1. } d(x) \sqcap y \approx x \sqcap y;$$

$$\text{S'2. } x \sqcap y \approx y \sqcap x;$$

$$\text{S'3. } (x \sqcap y) \sqcap z \approx x \sqcap (y \sqcap z);$$

$$\text{M'1. } (x \sqcap y)zz \approx (xzz) \sqcap (yzz);$$

$$\text{M'2. } (x \sqcap y)(x \sqcap z) \approx d(x) (x \sqcap (d(y)d(z)));$$

$$\text{M'3. } x \sqcap (xyy) = d(x).$$

The variety of all such algebras is denoted by \mathcal{V}_{EADS} .

As before, we have chosen the names of the axioms to indicate their purpose. S'1-S'3 make $(d(A), \sqcap)$ a semilattice and M'1-M'3 describe how the semilattice structure mixes with regularization and equivalence.

Lemma 2.17. $(\leftrightarrow, \sqcap)$ -reducts of Heyting algebras are EADS.

Proof. In a Heyting algebra $(x \sqcap x)x = (d_{\mathcal{H}}(x) \wedge d_{\mathcal{H}}(x)) \leftrightarrow x = d_{\mathcal{H}}(x) \leftrightarrow x = r_{\mathcal{H}}(x)$, so the regularization defined using \sqcap is the same as $r_{\mathcal{H}}$. It follows that E1-E3 and R1-R4 must be true, since $(\leftrightarrow, r_{\mathcal{H}})$ -reducts of Heyting algebras are equivalential algebras with regularization. Because $d_{\mathcal{H}}$ is idempotent, we have S'1; because \wedge is a semilattice operation we have S'2 and S'3. To show that M'1-M'3 hold in \mathcal{H} , it is enough to show that they hold in Brouwerian semilattices with zero. Due to the local finiteness of \mathcal{BS}_0 we can perform the “lack of a counterexample” proof approach. Similarly to Lemma 1.92, we can argue that if an identity is not universally true, then it must be false in some finite subdirectly irreducible algebra, and there exist such elements that one side of the equality is 1 and the other is $*$.

If the left-hand side of M'1 is $*$, then by Corollary 1.67 it implies $x \sqcap y = *, z \in \{1, *\}$. Assuming that the right-hand side is $*$, we get $d_{\mathcal{H}}(xzz) \wedge d_{\mathcal{H}}(yzz) = *$. \mathcal{BS}_0 satisfy requirements of Theorem 1.65 so removing $*$ from the universe yields a subalgebra. Because of that, $d_{\mathcal{H}}(xzz) \wedge d_{\mathcal{H}}(yzz) = *$ implies $* \in \{d_{\mathcal{H}}(xzz), d_{\mathcal{H}}(yzz)\}$, which in turn implies $* \in \{xzz, yzz\}$. This is possible only for $z \in \{1, *\}$. Therefore, if either side is equal to $*$, then $z \in \{1, *\}$. This means that $(x \sqcap y)zz = x \sqcap y, xzz = x, yzz = y$ and both sides of the identity are equal. Hence, there is no counterexample to M'1.

If the left-hand side of M'3 is equal to $*$, then by a similar argument $* \in \{d_{\mathcal{H}}(x), d_{\mathcal{H}}(xyy)\}$, so $* \in \{x, xyy\}$, which in turn can only be possible when $x = *$. If this is the case, then $xyy = *yy$ is equal to 1 or $*$; and $d_{\mathcal{H}}(xyy)$ is either $d_{\mathcal{H}}(1) = 1$ or $d_{\mathcal{H}}(*)$. In both situations, the left-hand side simplifies to $d_{\mathcal{H}}(x)$, so the equality holds. If the left-hand side of M'3 is equal to 1, then $d_{\mathcal{H}}(x) = d_{\mathcal{H}}(xyy) = 1$, so the right-hand side is also 1.

If the left-hand side of M'2 is $*$, then $\{x \sqcap y, x \sqcap z\} = \{1, *\}$. This is only possible with $d_{\mathcal{H}}(x) = 1$ and $\{d_{\mathcal{H}}(y), d_{\mathcal{H}}(z)\} = \{1, *\}$, which simplifies the right-hand side to $1(1 \wedge d_{\mathcal{H}}(1*)) = d_{\mathcal{H}}(*)$. Because removing $*$ from the universe creates a subalgebra,

we must have $d_{\mathcal{H}}(*) = *$ or it will contradict $* \in \{d_{\mathcal{H}}(y), d_{\mathcal{H}}(z)\}$. Now assume that the left-hand side is 1 and the right-hand side is $*$. This implies $d_{\mathcal{H}}(x) \wedge d_{\mathcal{H}}(y) = d_{\mathcal{H}}(x) \wedge d_{\mathcal{H}}(z)$ and $d_{\mathcal{H}}(x) \in \{1, *\}$. Because $*$ is the second-largest element, we either get $d_{\mathcal{H}}(y) = d_{\mathcal{H}}(z)$ or $d_{\mathcal{H}}(x) = *$, $\{d_{\mathcal{H}}(y), d_{\mathcal{H}}(z)\} = \{1, *\}$. In the first case, the right-hand side is simplified to $d_{\mathcal{H}}(x)(x \sqcap 1) = 1$ and in the second case it becomes $d_{\mathcal{H}}(*) (d_{\mathcal{H}}(*) \wedge d_{\mathcal{H}}(1*)) = d_{\mathcal{H}}(*) d_{\mathcal{H}}(*) = 1$.

Hence, the three equalities are true in all finite Brouwerian semilattices with zero, so they are identities of \mathcal{BS}_0 and \mathcal{H} . \square

The mixing axioms were chosen to be “minimal” in the sense that removing any of them would yield a larger variety. Such a variety would, in particular, contain algebras that are not $(\leftrightarrow, \sqcap)$ -subreducts of Heyting algebras. To show this, we present the following examples.

Example 2.18. Take the Heyting algebra obtained from the chain $\{0, \frac{1}{2}, 1\}$ with natural ordering and take (A, \cdot) to be its equivalential reduct. If we define \sqcap by

$$x \sqcap y = \begin{cases} 1, & \text{if } x = y = 1, \\ \frac{1}{2}, & \text{otherwise,} \end{cases}$$

then M'1 is not satisfied. For example

$$(\frac{1}{2} \sqcap 0) \cdot 0 \cdot 0 = \frac{1}{2} \cdot 0 \cdot 0 = 0 \cdot 0 = 1 \neq \frac{1}{2} = 1 \sqcap 0 = (\frac{1}{2} \cdot 0 \cdot 0) \sqcap (0 \cdot 0 \cdot 0).$$

However, by directly checking, (A, \cdot, \sqcap) satisfies all other identities from the above definition.

Example 2.19. Similarly to Example 2.3, we consider a four-element algebra (A, \cdot, \sqcap) , such that (A, \cdot) is a Boolean group with neutral element 1, and (A, \sqcap) is a semilattice obtained from a chain with maximum element 1. Such an algebra fails to satisfy M'2 but satisfies M'1 and M'3.

Example 2.20. Similarly to Example 2.4, consider a Heyting algebra which is a product of two chains $\{1, 0\} \times \{1, *, 0\}$. We take its equivalential reduct (A, \cdot) and define a semilattice operation on dense elements such that $(1, 1)$ is the largest, $(0, 1)$ the smallest, and $(1, *), (0, *)$ incomparable. Finally, we extend \sqcap to $A \times A$ so

that $d((x, 0)) = (x, 1)$ and S'1-S'3 hold. The resulting algebra satisfies all identities except M'3.

Again, we want to introduce some more properties of EADS to make the following proofs easier.

Lemma 2.21. The following identities are satisfied in \mathcal{V}_{EADS}

$$U'1 \quad d(x) \approx 1 \sqcap x;$$

$$U'2 \quad d(x) \approx r(x)x;$$

$$U'3 \quad x \approx d(x)r(x);$$

$$U'4 \quad x \sqcap y \approx d(x \sqcap y);$$

$$U'5 \quad r(d(x)) \approx 1;$$

$$U'6 \quad d(r(x)) \approx 1;$$

$$U'7 \quad d(x)r(xy)r(xy) \approx d(x)r(y)r(y);$$

$$U'8 \quad d(xy) \approx d(x)r(y)r(y)(d(y)r(x)r(x));$$

$$U'9 \quad d(x)yy \approx d(d(x)yy).$$

Proof. The proof is a simple application of identities. Let $\mathbf{A} \in \mathcal{V}_{EADS}$, $x, y, z \in \mathbf{A}$, we have the following.

Ad U'1)

$$1 \stackrel{E4}{=} (x \sqcap y)(x \sqcap y) \stackrel{M'2}{=} d(x)(x \sqcap d(y)d(y)) \stackrel{E4}{=} d(x)(x \sqcap 1).$$

equivalence of two elements is equal to 1 if and only if they are equal, so $d(x) = x \sqcap 1$.

Ad U'2)

$$r(x)x = d(x)xx = (x \sqcap x)xx \stackrel{M'1}{=} (xxx) \sqcap (xxx) \stackrel{E1}{=} x \sqcap x = d(x).$$

Ad U'3)

$$x \stackrel{RA}{=} xr(x)r(x) = d(x)r(x).$$

Ad U'4)

$$x \sqcap y \stackrel{S'1}{=} d(x) \sqcap y \stackrel{S'1}{=} d(x) \sqcap d(y) = x \sqcap x \sqcap y \sqcap y \stackrel{S'3}{=} (x \sqcap y) \sqcap (x \sqcap y) = d(x \sqcap y).$$

Ad U'5)

$$\begin{aligned} r(d(x)) &\stackrel{R3}{=} r(d(x))d(x)d(x) = d(d(x))d(x) \\ &= d(x \sqcap x)d(x) \stackrel{U'4}{=} (x \sqcap x)d(x) = d(x)d(x) \stackrel{E4}{=} 1. \end{aligned}$$

Ad U'6)

$$d(r(x)) \stackrel{U'2}{=} r(r(x))r(x) \stackrel{R1}{=} r(x)r(x) \stackrel{E4}{=} 1.$$

Ad U'7)

$$\begin{aligned} d(x)r(xy)r(xy) &\stackrel{U'2}{=} xr(x)r(xy)r(xy) \stackrel{U'3}{=} d(x)r(x)r(x)r(xy)r(xy) \\ &\stackrel{R2}{=} d(x)r(x)r(x)(r(x)r(y))(r(x)r(y)) \stackrel{E9}{=} d(x)r(x)r(x)r(y)r(y) \\ &\stackrel{U'3}{=} xr(x)r(y)r(y) \stackrel{U'2}{=} d(x)r(y)r(y). \end{aligned}$$

Ad U'8)

$$\begin{aligned} d(xy) &\stackrel{U'2}{=} xy r(xy) \stackrel{R7}{=} (xr(x))(yr(y))r(xy)r(xy) \\ &\stackrel{E10}{=} d(x)r(xy)r(xy)(d(y)r(xy)r(xy)) \stackrel{U'7}{=} d(x)r(y)r(y)(d(y)r(x)r(x)). \end{aligned}$$

Ad U'9)

$$\begin{aligned} d(d(x)yy) &\stackrel{U'2}{=} d(x)yyr(d(x)yy) \stackrel{R2}{=} d(x)yy(r(d(x))r(y)r(y)) \\ &\stackrel{R3}{=} d(x)yyr(d(x)) \stackrel{U'5}{=} d(x)yy1 \stackrel{E1}{=} d(x)yy. \end{aligned}$$

□

We would like to point out, that all the above properties except U'1, U'4 are true in equivalential algebras with regularization with $d(x) = xr(x)$. This is because \mathcal{E}_r are subreducts of EADS.

Instead of a list of identities, we may also say something more descriptive about the structure of algebras. As S'1-S'3 impose a semilattice structure on the dense part of the universe, we have an associated partial ordering \leq_d of dense elements.

Lemma 2.22. Let $\mathbf{A} = (A, \cdot, \sqcap)$ be an EADS. The following are true:

- (a) Function $d : A \mapsto A$ is idempotent;
- (b) $(d(A), \cdot, \sqcap)$ is a subalgebra of \mathbf{A} ;
- (c) $(d(A), \sqcap, 1)$ is a semilattice with the largest element 1. If $y, z \in d(A)$, then $y \cdot z$ is the largest (with respect to \leq_d) element $x \in d(A)$, such that $x \sqcap y = x \sqcap z$;
- (d) if $x \in d(A), y \in A$, then $x \leq_d xyy$.

Proof. (a) is easy to see by applying U'9 with $y = 1$. (b): By U'4, the image of \sqcap is contained in $d(A)$. Closedness under \cdot is guaranteed by Lemma 1.93, because $(A, \cdot, r) \in \mathcal{E}_r$. (c): If $y, z \in d(A)$, then

$$((yz) \sqcap y)((yz) \sqcap z) \stackrel{M'2}{=} d(yz)((yz) \sqcap (d(y)d(z))) = (yz)((yz) \sqcap (yz)) = (yz)(yz) = 1,$$

so $(yz) \sqcap y = (yz) \sqcap z$. Now take any $x \in d(A)$ such that $x \sqcap y = x \sqcap z$, then

$$1 = (x \sqcap y)(x \sqcap z) \stackrel{M'2}{=} d(x)(x \sqcap (yz)) = x(x \sqcap (yz)),$$

so $x = x \sqcap (yz), x \leq_d (yz)$ as intended.

Finally, (d) follows by a direct application of M'3. □

Now we would like to describe the congruences of EADS.

Lemma 2.23. Let $\mathbf{A} = (A, \cdot, \sqcap) \in \mathcal{V}_{EADS}$, then for any $x, w \in A$ we have

- (a) $\Theta_{\mathbf{A}}(1, x) = \Theta_{\mathbf{A}}(1, r(x)) \vee \Theta_{\mathbf{A}}(1, d(x))$;
- (b) $\Theta_{\mathbf{A}}(1, x) \subset \{(y, z) : r(z)r(y) \in \{1, r(x)\}\}$;
- (c) If $x = d(x)$, then $\Theta_{\mathbf{A}}(1, x) = \{(y, z) : y \sqcap x = z \sqcap x, r(y) = r(z)\}$;
- (d) If $\Theta_{\mathbf{A}}(1, d(x)) = \Theta_{\mathbf{A}}(1, d(w))$, then $d(x) = d(w)$;
- (e) $\Theta_{\mathbf{A}}(1, r(x)) = \{(y, z) : yzr(x)r(x) \in \{1, r(x)\}\}$.

Proof. (a): On the one hand, $(1, r(x)) \in \Theta_{\mathbf{A}}(1, x)$, $(1, d(x)) \in \Theta_{\mathbf{A}}(1, x)$, so

$$\Theta_{\mathbf{A}}(1, x) \supset \Theta_{\mathbf{A}}(1, r(x)) \vee \Theta_{\mathbf{A}}(1, d(x)).$$

On the other hand,

$$x = d(x)r(x) \equiv_{\Theta_{\mathbf{A}}(1, r(x))} d(x) \equiv_{\Theta_{\mathbf{A}}(1, d(x))} 1,$$

so $\Theta_{\mathbf{A}}(1, x) \subset \Theta_{\mathbf{A}}(1, r(x)) \vee \Theta_{\mathbf{A}}(1, d(x))$.

(b): We will show that, for a fixed $x \in A$, the set $C_x = \{(y, z) : r(z)r(y) \in \{1, r(x)\}\}$ is a congruence. For regular elements, the operation \cdot is associative, so C_x can be rewritten as $C_x = \{(y, z) : r(z) = r(y)\} \cup \{(y, z) : r(z) = r(x)r(y)\}$. It is easy to see that it is an equivalence relation. For $(y, z) \in C_x, w \in A$ we have

$$r(z) = r(y) \Rightarrow r(zw) \stackrel{R2}{=} r(z)r(w) = r(y)r(w) \stackrel{R2}{=} r(yw),$$

$$\begin{aligned} r(z) = r(x)r(y) &\Rightarrow r(zw) \stackrel{R2}{=} r(z)r(w) \\ &= r(x)r(y)r(w) \\ &\stackrel{R2}{=} r(xyw) \\ &\stackrel{R3}{=} r(xyw)r(y)r(y) \\ &\stackrel{R2}{=} r(x)r(y)r(w)r(y)r(y) \\ &\stackrel{E2}{=} r(x)r(y)r(y)(r(w)r(y)) \\ &\stackrel{R3}{=} r(x)(r(w)r(y)) \\ &\stackrel{R2}{=} r(x)r(wy) \end{aligned}$$

$$r(x \sqcap w) \stackrel{U'4}{=} r(d(x \sqcap w)) \stackrel{U'5}{=} 1 \stackrel{U'5}{=} r(d(y \sqcap w)) \stackrel{U'4}{=} r(y \sqcap w).$$

This shows, that $C_x \in \text{Con}(\mathbf{A})$. Because $r(x) = r(x)r(1)$ we have $(1, x) \in C_x$, so $\Theta_{\mathbf{A}}(1, x) \subset C_x$.

(c): Fix $x \in A, x = d(x)$. Consider the set $C_x = \{(y, z) : y \sqcap x = z \sqcap x, r(y) = r(z)\}$, which obviously is an equivalence relation. By M'2 and the properties of equivalential algebras, the condition $y \sqcap x = z \sqcap x$ can be rewritten as $x \sqcap (d(y)d(z)) = x$. Using the partial order of dense elements, this is the same as $x \leq_d (d(y)d(z))$.

Let $(y, z) \in C_x, w \in A$ and define $w' = d(w)r(y)r(y) = d(w)r(z)r(z)$ we have

$$\begin{aligned} (d(yw)d(zw)) &\stackrel{U'8}{=} ((d(y)r(w)r(w)w')(d(z)r(w)r(w)w')) \\ &\stackrel{E2}{=} (d(y)r(w)r(w)(d(z)r(w)r(w))w'w') \\ &\stackrel{E10}{=} (d(y)d(z)r(w)r(w)w'w'). \end{aligned}$$

Using the assumption that $(y, z) \in C_x$ and point (d) of the previous lemma, we have

$$x \leq_d d(y)d(z) \leq_d d(y)d(z)r(w)r(w) \leq_d d(y)d(z)r(w)r(w)w'w'$$

so $x \leq_d d(yw)d(zw), (yw, zw) \in C_x$.

To show $(y \sqcap w, z \sqcap w) \in C_x$ we use the associativity of \sqcap (identity S'3) and the equality

$$r(y \sqcap w) \stackrel{U'4}{=} r(d(y \sqcap w)) \stackrel{U'5}{=} 1 \stackrel{U'5}{=} r(d(z \sqcap w)) \stackrel{U'4}{=} r(z \sqcap w).$$

This implies $C_x \in \text{Con}(\mathbf{A})$. Because $x = d(x)$ we have $r(x) = 1 = r(1)$, so $(1, x) \in C_x$ and $\Theta_{\mathbf{A}}(1, x) \subset C_x$.

To show the inverse inclusion, take any $(y, z) \in C_x$. We have

$$\begin{aligned} z &\stackrel{U'3}{=} r(z)d(z) \\ &\stackrel{U'1}{=} r(z)(z \sqcap 1) \\ &\equiv_{\Theta(1,x)} r(z)(z \sqcap x) \\ &= r(y)(y \sqcap x) \\ &\equiv_{\Theta(1,x)} r(y)(y \sqcap 1) \\ &\stackrel{U'1}{=} r(y)d(y) \\ &\stackrel{U'3}{=} y, \end{aligned}$$

so $y \equiv_{\Theta(1,x)} z$, which implies $C_x \subset \Theta(1, x)$. □

(d): If $\Theta_{\mathbf{A}}(1, d(w)) = \Theta_{\mathbf{A}}(1, d(x))$, then by point (c) $1 \sqcap d(x) = d(w) \sqcap d(x)$ and $1 \sqcap d(w) = d(x) \sqcap d(w)$, so $d(w) \stackrel{U'1}{=} 1 \sqcap d(w) = 1 \sqcap d(x) \stackrel{U'1}{=} d(x)$.

(e): Fix $x \in A$. By Lemma 1.91 we see that $C_x = \{(y, z) : yzr(x)r(x) \in \{1, r(x)\}\}$ is an equivalence relation that preserves the operation \cdot ; we need to check,

it preserves \sqcap . Take $(y, z) \in C_x, w \in A$. If $yzr(x)r(x) = 1$, then

$$\begin{aligned} yr(x)r(x) &= zr(x)r(x), \\ d(yr(x)r(x)) &= d(zr(x)r(x)), \\ d(y)r(x)r(x) &= d(z)r(x)r(x), \\ d(y)d(z)r(x)r(x) &= 1. \end{aligned}$$

On the other hand, if $yzr(x)r(x) = r(x)$, then

$$\begin{aligned} r(yzr(x)r(x)) &= r(r(x)), \\ r(y)r(z)r(r(x))r(r(x)) &= r(r(x)), \\ r(y)r(z) &= r(x). \end{aligned}$$

By associativity of equivalence on regular elements, $r(z) = r(xy), r(y) = r(xz)$, and

$$\begin{aligned} 1 &\stackrel{U'6}{=} d(r(x)) \\ &= d(yzr(x)r(x)) \\ &\stackrel{U'9}{=} d(yz)r(x)r(x) \\ &\stackrel{U'8}{=} (d(y)r(z)r(z))(d(z)r(y)r(y))r(x)r(x) \\ &\stackrel{E10}{=} (d(y)r(z)r(z)r(x)r(x))(d(z)r(y)r(y)r(x)r(x)) \\ &= (d(y)r(xy)r(xy)r(x)r(x))(d(z)r(xz)r(xz)r(x)r(x)) \\ &\stackrel{E9}{=} (d(y)r(y)r(y)r(x)r(x))(d(z)r(z)r(z)r(x)r(x)) \\ &\stackrel{E10}{=} (d(y)r(y)r(y))(d(z)r(z)r(z))r(x)r(x) \\ &\stackrel{M'1}{=} d(yr(y)r(y))d(zr(z)r(z))r(x)r(x) \\ &\stackrel{R4}{=} d(y)d(z)r(x)r(x). \end{aligned}$$

Therefore, in both cases we have

$$\begin{aligned}
(y \sqcap w)(z \sqcap w)r(x)r(x) &\stackrel{M'2}{=} d(w)(w \sqcap (d(y)d(z)))r(x)r(x) \\
&\stackrel{E10}{=} d(w)r(x)r(x)((w \sqcap (d(y)d(z)))r(x)r(x)) \\
&\stackrel{M'1}{=} d(wr(x)r(x))((wr(x)r(x)) \sqcap (d(y)d(z)r(x)r(x))) \\
&= d(wr(x)r(x))((wr(x)r(x)) \sqcap 1) \\
&\stackrel{U'1}{=} d(wr(x)r(x))d(wr(x)r(x)) \\
&= 1.
\end{aligned}$$

This shows that C_x is a congruence, and of course $(1, r(x)) \in C_x$. For any pair $(y, z) \in C_x$, we have

$$\begin{aligned}
y &\stackrel{U'3}{=} r(y)d(y) \\
&\equiv_{\Theta_{\mathbf{A}}(1, r(x))} r(z)(d(y)r(x)r(x)) \\
&= r(z)(d(z)r(x)r(x)) \\
&\equiv_{\Theta_{\mathbf{A}}(1, r(x))} r(z)d(z) \stackrel{U'3}{=} z,
\end{aligned}$$

showing that $\Theta_{\mathbf{A}}(1, r(x)) = C_x$.

Theorem 2.24. *The variety \mathcal{V}_{EADS} is congruence permutable and Fregean.*

Proof. As in the case of EARS, we only need to show congruence orderability. Consider any $\mathbf{A} = (A, \cdot, \sqcap) \in \mathcal{V}_{EADS}$ and $a, b \in A$ such that $\Theta_{\mathbf{A}}(1, a) = \Theta_{\mathbf{A}}(1, b)$. By point (b) of the previous lemma

$$(1, b) \in \Theta_{\mathbf{A}}(1, a) \subset \{(y, z) : r(z)r(y) \in \{1, r(a)\}\},$$

so $r(b) = 1$ or $r(b) = r(a)$. Symmetrically, $r(a) = 1$ or $r(a) = r(b)$. It follows that $r(a) = r(b)$ must hold. Let $\alpha = \Theta_{\mathbf{A}}(1, r(a)) = \Theta_{\mathbf{A}}(1, r(b))$, by Lemma 2.23 point (a)

$$\Theta_{\mathbf{A}}(1, d(a)) \vee \alpha = \Theta_{\mathbf{A}}(1, a) = \Theta_{\mathbf{A}}(1, b) = \Theta_{\mathbf{A}}(1, d(b)) \vee \alpha,$$

$$\Theta_{\mathbf{A}/\alpha}(1/\alpha, d(a/\alpha)) = \Theta_{\mathbf{A}/\alpha}(1/\alpha, d(b/\alpha)).$$

Now, by point (d) of the aforementioned lemma, $d(a) \equiv_{\alpha} d(b)$ and by point (e)

$$abr(a)r(a) \stackrel{E2}{=} ar(a)(br(a)) = d(a)d(b) \in \{1, r(a)\}.$$

If $d(a)d(b) = r(a)$, then $r(a) \stackrel{R1}{=} r(d(a)d(b)) \stackrel{R2}{=} r(d(a))r(d(b)) \stackrel{U'5}{=} 1$, so we get $d(a) = d(b), a = b$. \square

Lemma 2.25. All nontrivial $\mathbf{A} \in \mathcal{V}_{EADS}$ have unique extensions.

Proof. Let \mathbf{B} be subdirectly irreducible such that $\mathbf{B}/\mu_{\mathbf{B}} \simeq \mathbf{A}$. By Lemma 1.68, \mathbf{B} can be obtained by adding to \mathbf{A} a new element $*$ and extending the operations in such a way that the pair $(1, *)$ generates the monolith. By Corollary 1.88, $r(*) = 1$, so $d(*) = *$, and the equivalence operation can be extended in only one way because of the unique extensions property of \mathcal{E} . Because $*\sqcap x = *\sqcap d(x)$, we only need to show that there is a unique way of adding $*$ to the partial ordering of dense elements.

If for any $x \notin \{1, *\}$ the operation \sqcap is such that $x\sqcap * \in \{*, 1\}$, then $x \in 1/\Theta(1, *)$; on the other hand if x and $*$ are incomparable, then x and $x\sqcap *$ are two distinct elements, but $x \equiv_{\Theta(1, *)} x\sqcap *$. Both cases contradict the fact that $\Theta(1, *)$ is the monolith, as from Lemma 1.62 we know that the only nontrivial class of the monolith is $\{1, *\}$. Hence, the only way is to make $*$ larger than any element other than 1, this means, the operation \sqcap can only be properly extended by taking

$$*\sqcap x = \begin{cases} * & \text{if } x = 1, \\ x & \text{otherwise.} \end{cases}$$

This makes \mathbf{B} unique, by direct verification the algebra obtained in this way is an EADS. \square

Lemma 2.26. There are two simple algebras in \mathcal{V}_{EADS} (up to isomorphism); both are $(\leftrightarrow, \sqcap)$ -subreducts of Heyting algebras.

Proof. By Corollary 1.64, a simple EADS must be two-element. As in \mathcal{V}_{EARS} and \mathcal{E}_r , the equivalence is determined uniquely, while there are two ways to define the other operation. One possibility is $r(*) = 1$, then \sqcap is the minimum operation, and the algebra is a reduct of a two-element Heyting chain. The other option is $r(*) = *$, then \sqcap is always 1 and the algebra is a subreduct of a three-element chain. \square

Remark 2.27. Of these two simple algebras, the first one is polynomially equivalent to a two-element Boolean algebra, so it is of type **3**. In the second one, \sqcap does not

introduce any new polynomial, so the algebra is polynomially equivalent to the two-element equivalential algebra with regularization, which is of type **2**. Therefore, any EADS that contains both of these as subalgebras is of mixed type. The simplest example is the $(\leftrightarrow, \sqcap)$ -reduct of the three-element Heyting chain; this is the algebra **D** investigated by Przybyło.

Theorem 2.28. *The variety \mathcal{V}_{EADS} is the class of all $(\leftrightarrow, \sqcap)$ -subreducts of Heyting algebras.*

Proof. Identities M'1 and E10 guarantee that the algebras in \mathcal{V}_{EADS} satisfy the condition (\dagger) from Theorem 1.65, so the class is locally finite and primitive. Let \mathcal{S} be the class of all $(\leftrightarrow, \sqcap)$ -subreducts. We have $\mathcal{S} \subset \mathcal{V}_{EADS}$ because all the identities required by the definition of EADS are true in Heyting algebras. By the primitivity of \mathcal{V}_{EADS} , \mathcal{S} is a variety. By Theorem 2.15, all finitely generated members of \mathcal{V}_{EADS} are in \mathcal{S} , so the varieties must be equal. \square

Chapter 3

Searching for other reducts

In the previous chapter, we have shown that two classes of Heyting algebra subreducts are congruence permutable Fregean algebras. Of course, those are not the only such classes. Other examples include $\mathcal{E}, \mathcal{E}_r, \mathcal{BS}, \mathcal{BS}_0$. On the other hand, as shown in [23], the (\leftrightarrow, \neg) -subreducts are not a variety but a quasivariety. This happens because using identities only cannot guarantee that, for nontrivial algebras, 1 and 0 are different constants. In fact, to describe this class one only needs four identities and one simple quasi-identity

$$0 \approx 1 \rightsquigarrow x \approx 1. \tag{q01}$$

The reason why we wanted to investigate \mathcal{V}_{EARS} and \mathcal{V}_{EADS} is that, unlike the mentioned examples, they contain mixed type algebras in the meaning of Tame Congruence Theory [12]. In this chapter, we search whether there are other such classes among subreducts of Heyting algebras.

Research question: Let \mathcal{G} be a family of terms in Heyting algebras such that the class of all \mathcal{G} -subreducts of Heyting algebras is congruence permutable Fregean. When such a class is of mixed type?

We do not have a complete answer to the above question, but we will be able to give a partial answer under some additional assumptions. All congruence permutable Fregean varieties have a term acting as equivalence, so we require \leftrightarrow to be in the

reduced language. Moreover, the tools used previously demand that our candidate class meets the requirements of Theorem 1.65. The only basic operation in Heyting algebras that does not satisfy the condition (\dagger) is \vee , so we restrict ourselves to terms that can be written using only $(\rightarrow, \wedge, 0)$. Moreover, we will focus on binary terms only.

While showing congruence orderability of EARS and EADS we used the fact that the unary regularization lets us split the congruences into two parts using the equation

$$\Theta_{\mathbf{A}}(1, a) = \Theta_{\mathbf{A}}(1, d(a)) \vee \Theta_{\mathbf{A}}(1, r(a)).$$

This is the reason why we will split our investigation into two main cases depending on whether the clone generated by (\leftrightarrow, t) contains $r_{\mathcal{H}}$.

If we consider an algebra \mathbf{A} , which is a (\leftrightarrow, t) -reduct of $\mathbf{H} \in \mathcal{BS}_0$, then $\text{Clo}(\mathbf{A})$ is the same (as a set of functions) as the clone in \mathbf{H} generated by the pair $\{\leftrightarrow, t\}$. If the pairs of terms $\{\leftrightarrow, t\}$ and $\{\leftrightarrow, u\}$ generate the same clone in \mathbf{H} , then the respective reducts will have the same clones of terms. Hence, each (\leftrightarrow, t) -subreduct will be term equivalent to a (\leftrightarrow, u) -subreduct with the same universe. For example, take $t(x, y) = \neg\neg x \rightarrow \neg\neg y$. In Heyting algebras, it is true that $x \widehat{\wedge} y = t(x, y) \leftrightarrow t(1, x)$ and $t(x, y) = (x \widehat{\wedge} y) \leftrightarrow (x \widehat{\wedge} x)$, so $\{\leftrightarrow, t\}$ generates the same clone as $\{\leftrightarrow, \widehat{\wedge}\}$ (to check that remember, that by Remark 1.34 the subset of regular elements of any Heyting algebra form a Boolean algebra with the same \wedge and \rightarrow operations). Therefore, the class of (\leftrightarrow, t) -subreducts is of mixed type because each its member is term equivalent to an EARS and vice versa.

This is why we will focus on \mathcal{G} -subreducts of \mathcal{BS}_0 , where \mathcal{G} is not an arbitrary family, but a “maximal” one in the sense that it forms a clone when interpreted in every Brouwerian semilattice with zero. More precisely, for any $\mathbf{A} \in \mathcal{BS}_0$ we want the set $\{t^{\mathbf{A}} : t \in \mathcal{G}\}$ to be a subclone of $\text{Clo}_2(\mathbf{A})$. But, since we are working with binary terms only, this is equivalent to $\{t^{\mathbf{F}_2} : t \in \mathcal{G}\}$ being a subclone of $\text{Clo}(\mathbf{F}_2)$ (we remind that \mathbf{F}_2 is the free Brouwerian semilattice over two generators). Later, for any pair $\{\leftrightarrow, t\}$, we can just check what set of terms it generates and conclude if the respective subreducts have a mixed type. We want to know when such a \mathcal{G} -subreduct is a class containing mixed type algebras.

In Section 1.5, we presented the construction of the algebra \mathbf{F}_2 (we use x, y to denote generators). Since we are working with a free algebra, there is a one-to-one correspondence between a term $t \in T_{\{\rightarrow, \wedge, 0\}}(x, y)$, a term operation $t^{\mathbf{F}_2} \in \text{Clo}_2(\mathbf{F}_2)$ and an element $t^{\mathbf{F}_2}(x, y) \in \mathbf{F}_2$. To every $\mathcal{G} \subset T_{\{\rightarrow, \wedge, 0\}}(x, y)$, we assign $G_{\mathcal{G}} = \{t^{\mathbf{F}_2}(x, y), t \in \mathcal{G}\} \subset \mathbf{F}_2$. We look for such \mathcal{G} , that $x, y \in G_{\mathcal{G}}$, and for any $t_1, t_2, t_3 \in \mathcal{G}$, $t_1^{\mathbf{F}_2}(t_2^{\mathbf{F}_2}(x, y), t_3^{\mathbf{F}_2}(x, y)) \in G_{\mathcal{G}}$. Of course $G_{\mathcal{G}}$ can be identified with a subclone of $\text{Clo}(\mathbf{F}_2)$.

The test algebra \mathbf{T}_7 was also introduced in the aforementioned section. Per Lemma 1.83, two binary terms in \mathbf{F}_2 are equal if they are equal as functions on \mathbf{T}_7 . The “interpretation” function $i : \mathbf{F}_2 \mapsto \text{Univ } \mathbf{T}_7^{\text{Univ } \mathbf{T}_7 \times \text{Univ } \mathbf{T}_7}$ assigns to each term the respective binary function; $i(u) = u^{\mathbf{T}_7}$. Take $\mathcal{G} \subset T_{\{\rightarrow, \wedge, 0\}}(x, y)$. If in \mathbf{T}_7 we have a function $v(a, b) = u_1^{\mathbf{T}_7}(u_2^{\mathbf{T}_7}(a, b), u_3^{\mathbf{T}_7}(a, b))$ for some $u_1, u_2, u_3 \in \mathcal{G}$, then $i(u_1(u_2, u_3)) \in i(G_{\mathcal{G}})$. This shows that $i(G_{\mathcal{G}})$ is closed under compositions; it obviously also contains projections. By the Lemma 1.83, i is injective. If we restrict the codomain to $i(\mathbf{F}_2)$, it will be a bijection between terms and certain functions on \mathbf{T}_7 . This lets us use computer assistance to perform an initial investigation and formulate the main result that we will formally show in the following sections.

We concluded that there are 12 subsets of $i(\mathbf{F}_2)$ that are closed under compositions, contain projections, equivalence, and regularization. These correspond to 12 subclones of $\text{Clo}(\mathbf{F}_2)$, which contain equivalence and regularization. Ten of them can be generated by a pair $\{\leftrightarrow, t\}$ for some binary t (the remaining two are generated by \leftrightarrow and two more terms). In total, 1556 out of 2134 binary terms t , are such that the clone generated by $\{\leftrightarrow, t\}$ is one of those ten possibilities. If we order those clones by inclusion, then we obtain the structure depicted in Figure 3.1.

Of those 12 clones, we have already considered the two corresponding to EARS and EADS. The largest clone is the full clone, which contains all binary terms of \mathcal{BS}_0 ; the clones generated by $\{\leftrightarrow, r\}$ and $\{\leftrightarrow, \neg\}$ correspond to equivalential algebras with regularization and negation, respectively. This leaves only 7 clones to consider. Notice that for any two terms t, u the supremum of the clones generated by $\{\leftrightarrow, u\}, \{\leftrightarrow, t\}$ is the clone generated by $\{\leftrightarrow, u, t\}$. This is why the most interesting of the remaining cases will be the clone generated by $\{\leftrightarrow, x(\neg y)(\neg y)\}$.

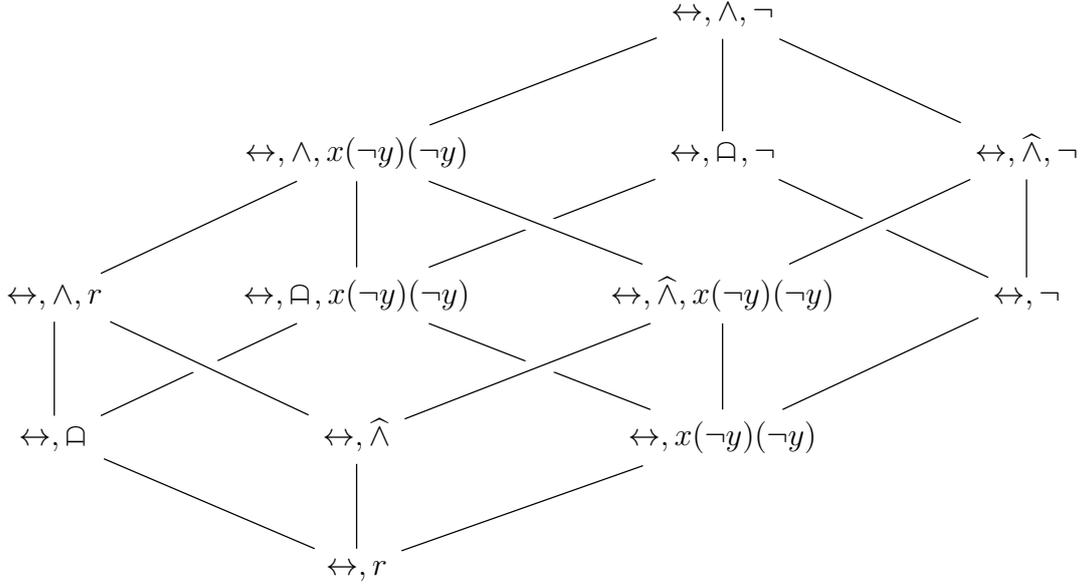


Figure 3.1: The sublattice of clones on \mathbf{F}_2 which contain equivalence and regularization. A label of each clone is an example set of generators.

From now on, we use the symbol $-$ to denote a binary operation; $x - y$ is a shorthand for $x(\neg y)(\neg y)$.

For a moment let us focus on reducts instead of subreducts. Consider any Heyting algebra \mathbf{H} and its equivalential reduct \mathbf{E} . Each of the terms $r, -, \neg$ can be written as a polynomial in \mathbf{E} , because the constant 0 is an element of the algebra. Therefore, (\leftrightarrow, r) -reduct, $(\leftrightarrow, -)$ -reduct, and (\leftrightarrow, \neg) -reduct of \mathbf{H} are all polynomially equivalent to \mathbf{E} . We can think of the terms $r, -, \neg$ as “stages of adding 0 to the language”. Looking back at the figure, the clones are arranged into four three-element chains. In each chain, the respective reducts have the same clones of polynomials by the same argument as above (terms from the larger language can be written as a polynomial in the smaller language). Therefore, if \mathcal{G} is one of the 12 clones containing regularization, then \mathcal{G} -reduct of a Heyting algebra is polynomially equivalent to exactly one of EARS, EADS, equivalential algebra, or a Brouwerian semilattice. Polynomial equivalence in particular means that the algebras have the same congruences and commutator, so they are of the same type. If we start with a three-element Heyting algebra, then the EARS and EADS reducts are of mixed type. On the other hand, Brouwerian semilattices are of type **3** and equivalential algebras are of type **2**. Therefore, we expect half of the 12 classes to contain mixed

type algebras, and the other half to have constant types.

There are also about 30 clones on \mathbf{F}_2 , which contain equivalence but not regularization. Later we will show that none of them give rise to mixed type reducts. The main result of this dissertation is the following.

Theorem 3.1. *Let $\mathcal{G} \subset T_{\{\rightarrow, \wedge, 0\}}(x, y)$. If the class of \mathcal{G} -subreducts of \mathcal{BS}_0 is a congruence permutable Fregean of mixed type, then it is one of six possibilities. Five of those classes are varieties, one is a quasivariety. As a bonus, we know the (quasi-)identities defining those classes.*

As mentioned earlier, \mathcal{G} -subreducts can be congruence permutable Fregean only if $\leftrightarrow \in \mathcal{G}$, so we will just assume that this condition is true from now on.

The above theorem is not a full answer to our question, because it is restricted only to a binary case and addresses the subreducts of \mathcal{BS}_0 instead of \mathcal{H} . We are planning to continue the research in this field in the future. The remainder of this work constitutes a proof of the above theorem. First, we will show that there are at most six such clones and then describe the remaining four of them as a (quasi-)variety.

3.1 Characterizing clones

In this section, we will investigate \mathbf{F}_2 using the representation of this algebra as $\text{Up}(\text{Cm}(\mathbf{F}_2), \wedge, \rightarrow, 0)$, as described in Section 1.5. We will use K to denote the universe of $\text{Up}(\text{Cm}(\mathbf{F}_2))$. The set K has a natural structure of a Brouwerian semilattice with

$$\begin{aligned} a \rightarrow b &= ((a \setminus b) \downarrow)', \\ a \wedge b &= a \cap b, \\ 0 &= \emptyset \end{aligned}$$

for $a, b \in K$. It is well known that the function $M : \mathbf{A} \ni a \mapsto \{\varphi \in \text{Cm}(A) : \varphi \geq \Theta_{\mathbf{A}}(1, a)\}$ is a Brouwerian semilattice isomorphism for any finite \mathbf{A} . It also preserves zero, and as mentioned in Theorem 1.84 it preserves \leftrightarrow . We recall that \leftrightarrow is defined

on K by

$$a \leftrightarrow b = ((a \div b) \downarrow)'$$

As $M(0) = \emptyset$, we can also define regularization on K by:

$$r(a) = (a \leftrightarrow \emptyset) \leftrightarrow \emptyset = (((a \downarrow)') \downarrow)'$$

In particular, M is a \mathcal{E}_r -isomorphism between (\cdot, r) -reduct of \mathbf{F}_2 and (K, \leftrightarrow, r) . The neutral element in (K, \leftrightarrow, r) is $1 = M(1) = \text{Cm}(\mathbf{F}_2)$.

We want to warn the reader that most of the following results are proven by reducing an algebraic problem to a combinatorial one and solving the latter, usually by brute calculation.

For simplicity, we name the elements of $\text{Cm}(\mathbf{F}_2)$ as depicted in Figure 3.2. The labels are introduced in such a way that $M(x) = \{a_3, a_4, b_6\}$, $M(y) = \{a_1, a_3, b_4\}$.

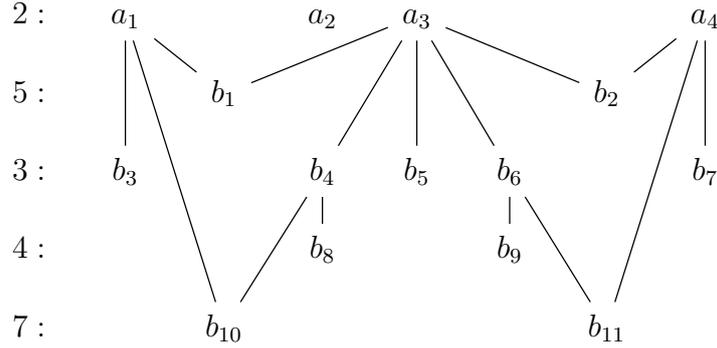


Figure 3.2: The poset $\text{Cm}(\mathbf{F}_2)$ with elements labeled.

The poset $\text{Cm}(\mathbf{F}_2)$ has an obvious automorphism, which is the permutation

$$(a_1, a_4)(b_1, b_2)(b_3, b_7)(b_4, b_6)(b_8, b_9)(b_{10}, b_{11}).$$

We introduce the symmetry map $s : K \mapsto K$, which assigns to a subset of $\text{Cm}(\mathbf{F}_2)$ its image under the aforementioned automorphism. Of course $s(M(x)) = M(y)$, $s(M(y)) = M(x)$. The following representations of elements of \mathbf{F}_2 will be useful later.

Lemma 3.2. The mapping M satisfies

$$\begin{aligned}
M(x) &= \{a_3, a_4, b_6\}, \\
M(y) &= \{a_1, a_3, b_4\}, \\
M(xy) &= \{a_2, a_3, b_5\}, \\
M(x \wedge y) &= \{a_3\}, \\
M(r(x)) &= \{a_3, a_4, b_2, b_4, b_5, b_6, b_7, b_8, b_9, b_{11}\}, \\
M(r(y)) &= \{a_1, a_3, b_1, b_3, b_4, b_5, b_6, b_8, b_9, b_{10}\}, \\
M(x \widehat{\wedge} y) &= \{a_3, b_4, b_5, b_6, b_8, b_9\}, \\
M(d(x)) &= \{a_1, a_2, a_3, a_4, b_1, b_3, b_6\}, \\
M(d(y)) &= \{a_1, a_2, a_3, a_4, b_2, b_4, b_7\}, \\
M(x \sqcap y) &= \{a_1, a_2, a_3, a_4\}, \\
M(\neg y) &= \{a_2, a_4, b_2, b_7\}, \\
M(x(\neg y)) &= \{a_1, a_4, b_3\}, \\
M(x - y) &= \{a_3, a_4, b_2, b_4, b_5, b_6, b_8, b_9, b_{11}\} = (\{a_1, a_2, b_7\} \downarrow)'.
\end{aligned}$$

Proof. We have chosen the labels in such a way that $M(x)$ and $M(y)$ are as indicated. For any $a, b \in \mathbf{F}_2$, we have $M(ab) = ((M(a) \div M(b)) \downarrow)'$, $M(a \wedge b) = M(a) \cap M(b)$ and $M(0) = \emptyset$. The rest of the proof is a direct computation. Another possible approach is to look at the construction of \mathbf{F}_2 . $M(t)$ is a subset of these $\varphi \in \text{Cm}(\mathbf{F}_2)$ such that $t \equiv_\varphi 1$. We just need to check in which labeled homomorphic images of \mathbf{F}_2 we have $t(x, y) = 1$. \square

First, we want to differentiate whether a clone of terms contains the regularization operation or not.

Lemma 3.3. Let C be a clone of binary terms in \mathbf{F}_2 generated by a pair $\{\leftrightarrow, t\}$. We have $r(x) \in C$ if and only if $\{t(x, x), t(1, x), t(x, 1)\} \not\subseteq \{1, x\}$.

Proof. There are only six unary terms in \mathbf{F}_2 , as described in the construction of the free algebra in Section 1.5 ($|\mathbf{F}_1| = 6$). These terms are: $0, 1, x, d(x), r(x), \neg x$. Hence, the condition $\{t(x, x), t(1, x), t(x, 1)\} \not\subseteq \{1, x\}$ can be rewritten as

$$\{t(x, x), t(1, x), t(x, 1)\} \cap \{d(x), r(x), \neg x, 0\} \neq \emptyset.$$

Now notice, that if $d(x) \in C$, then $r(x) = d(x) \leftrightarrow x \in C$; if $\neg x \in C$, then $r(x) = \neg\neg x \in C$; and if $0 \in C$, then $r(x) = (x \leftrightarrow 0) \leftrightarrow 0 \in C$.

On the other hand, if $\{t(x, x), t(1, x), t(x, 1)\} \subset \{1, x\}$, then $\{1, x\}$ is closed under operations \leftrightarrow, t . So $\{1, x\}$ must also be closed under every operation in the clone C . Hence, there cannot be any more unary terms in C , $r(x) \notin C$. \square

Now, we need a way to interpret the above result in K .

Lemma 3.4. For $t \in \mathbf{F}_2$, $t(x, x) = x$ if and only if $M(t) \cap \{a_2, a_3, b_5\} = \{a_3\}$.

Proof. If $t(x, x) = x$, then

$$t(x, y) \equiv_{\Theta(x, y)} t(x, x) = x,$$

so $\Theta(t(x, y), x) \subset \Theta(x, y)$. On the other hand, if $\Theta(t(x, y), x) \subset \Theta(x, y)$, then, by the universal mapping property, there exists a homomorphism $f : \mathbf{F}_2 \mapsto \mathbf{F}_2$, with $f(x) = f(y) = x$. Hence, $\Theta(x, y) \subset \ker f$ and $\Theta(t(x, y), x) \subset \ker f$, so we have an equality

$$t(x, x) = t(f(x), f(y)) = f(t(x, y)) = f(x) = x.$$

Therefore, $t(x, x) = x$ if and only if $\Theta(t(x, y), x) \subset \Theta(x, y)$.

As \cdot is the principal congruence term, $\Theta(t(x, y), x) \subset \Theta(x, y)$ is equivalent to $\Theta(t(x, y)x, 1) \subset \Theta(xy, 1)$, which is equivalent to $M(xy) \subset M(tx)$. In this way, we reduced a property of a term to simple combinatorics. From Lemma 3.2 we have $M(xy) = \{a_2, a_3, b_5\} = (\{a_1, a_4, b_4, b_6\} \downarrow)'$, therefore

$$\begin{aligned}
& M(xy) \subset M(tx) \\
& \Leftrightarrow M(xy)' \supset M(tx)' \\
& \Leftrightarrow \{a_1, a_4, b_4, b_6\} \downarrow \supset (M(t) \div M(x)) \downarrow \\
& \Leftrightarrow \{a_1, a_4, b_4, b_6\} \downarrow \supset M(t) \div M(x) \\
& \Leftrightarrow \{a_1, a_4, b_4, b_6\} \downarrow \supset M(t) \div \{a_3, a_4, b_6\} \\
& \Leftrightarrow (\{a_1, a_4, b_4, b_6\} \downarrow)' \subset (M(t) \div \{a_3, a_4, b_6\})' \\
& \Leftrightarrow \{a_2, a_3, b_5\} \subset (M(t) \div \{a_3, a_4, b_6\})' \\
& \Leftrightarrow \{a_2, a_3, b_5\} \subset (M(t) \cap \{a_3, a_4, b_6\}) \cup (M(t))' \cap \{a_3, a_4, b_6\}' \\
& \Leftrightarrow \{a_3\} \subset M(t) \text{ and } \{a_2, b_5\} \subset M(t)' \\
& \Leftrightarrow M(t) \cap \{a_2, a_3, b_5\} = \{a_3\}.
\end{aligned}$$

□

By the definition of M , we can look at the above result in another way. The inequality $t(x, x) \neq x$ holds if and only if we can find a “counterexample”: an algebra $\mathbf{A} \in \mathcal{BS}_0$ and homomorphism $h : \mathbf{F}_2 \mapsto \mathbf{A}$ with $h(x) = h(y)$ which does not satisfy $h(x) = h(t)$. In other words, gluing x with y does not necessarily imply gluing $t(x, y)$ with x . Moreover, we can assume without the loss of generality that the image of h is subdirectly irreducible (or we could decompose \mathbf{A} to find a smaller counterexample). To check whether $t(x, x) = x$, we look at the three completely meet-irreducible congruences on \mathbf{F}_2 that contain the pair (x, y) , and verify whether they are in $M(t)$ if and only if they are in $M(x)$.

Repetition of the above proof results in a similar characterization of all other properties of t needed to verify if the regularization is in the respective clone.

Lemma 3.5. For $t \in \mathbf{F}_2$ the following hold

$$\begin{array}{ll}
t(x, x) = x \Leftrightarrow M(xy) \subset M(tx) & \Leftrightarrow M(t) \cap \{a_2, a_3, b_5\} = \{a_3\}, \\
t(x, x) = 1 \Leftrightarrow M(xy) \subset M(t) & \Leftrightarrow M(t) \supset \{a_2, a_3, b_5\}, \\
t(x, 1) = x \Leftrightarrow M(y) \subset M(tx) & \Leftrightarrow M(t) \cap \{a_1, a_3, b_4\} = \{a_3\}, \\
t(x, 1) = 1 \Leftrightarrow M(y) \subset M(t) & \Leftrightarrow M(t) \supset \{a_1, a_3, b_4\}, \\
t(1, y) = y \Leftrightarrow M(x) \subset M(ty) & \Leftrightarrow M(t) \cap \{a_3, a_4, b_6\} = \{a_3\}, \\
t(1, y) = 1 \Leftrightarrow M(x) \subset M(t) & \Leftrightarrow M(t) \supset \{a_3, a_4, b_6\}, \\
t(1, 1) = 1 \Leftrightarrow M(x) \cap M(y) \subset M(t) & \Leftrightarrow M(t) \supset \{a_3\}.
\end{array}$$

Proof. The proof for each line is analogous to the previous lemma. In each case, we first show that the equation is equivalent to an inclusion between congruences, which is equivalent to an inclusion between members of K , which is finally interpreted as a property of $M(t)$. \square

Applying the above directly to Lemma 3.3, we get our desired characterization.

Corollary 3.6. *Let $C \subset \mathbf{F}_2$ be a clone of binary terms in the language \mathcal{BS}_0 generated by a pair $\{\leftrightarrow, t\}$. We have $r(x) \notin C$ if and only if $a_3 \in M(t)$ and each of the pairs $\{a_2, b_5\}, \{a_1, b_4\}, \{a_4, b_6\}$ is a subset of $M(t)$ or does not have a common element with it.*

Using the above, we can easily count the number of terms t such that the clone generated by $\{\leftrightarrow, t\}$ does not contain regularization. If we assume that $\{a_1, b_4\} \not\subset M(t)$, then $\{b_1, b_3, b_8, b_{10}\} \cap M(t) = \emptyset$, otherwise $\{b_1, b_3, b_8, b_{10}\} \cap M(t)$ may be arbitrary. A similar relation holds for pair $\{a_4, b_6\}$ and set $\{b_2, b_7, b_9, b_{11}\}$. Moreover, $M(t)$ either contains $\{a_2, b_5\}$ or has an empty intersection with it. Those three choices are independent of each other, so the number of possibilities sums up to $(1+2^4) \cdot (1+2^4) \cdot 2 = 578$, which agrees with our initial computer-assisted verification. Of the 2134 terms, 1556 do generate regularization and 578 do not.

For now, we will focus on the clones that contain regularization. A clone of binary terms in the language $\rightarrow, \wedge, \neg$ can be treated as a set $C \subset \mathbf{F}_2$, so we can consider its image $M(C) \subset K$. The closedness of a clone under the composition

means that

$$t_1, t_2, t_3 \in C \quad \Rightarrow \quad t_1^{\mathbf{F}_2}(t_2, t_3) \in C.$$

Therefore, if $C \subset \mathbf{F}_2$ is the set of elements that (when treated as functions) form a clone, then $M(C)$ must be closed under \leftrightarrow and r . Because the clone contains projections, we also have $M(x), M(y) \in M(C)$, and $M(C)$ must be closed under symmetric mapping s . Moreover, if $t \in C$, then $t^{\mathbf{F}_2}(C, C) \subset C$, in particular $t^{\mathbf{F}_2}(1, 1) \in C$. The value of $t^{\mathbf{F}_2}(1, 1)$ is equal to 1 or 0, and we can characterize when it is 0 using Lemma 3.5. In this way, we obtain some necessary conditions for a set $C \subset \mathbf{F}_2$ to be a clone with respect to the properties of $M(C) \subset K$.

Definition 3.7. A *clone candidate* is a set $C \subset \mathbf{F}_2$ such that:

- the generators x, y are members of C ;
- $M(C)$ is closed under operations \leftrightarrow, r, s ;
- if there exists $t \in C$ with $a_3 \notin M(t)$, then $\emptyset \in M(C)$.

We will now try to characterize all clone candidates, which will give us an upper bound for the number of clones (which contain equivalence and regularization). First, we need a way to simplify some calculations.

Lemma 3.8. For $u \in K, c \in \text{Cm}(\mathbf{F}_2)$ we have

$$c \in r(u) \Leftrightarrow (\{c\} \uparrow \cap \{a_1, a_2, a_3, a_4\}) \subset u,$$

$$r(u) = r(u \cap \{a_1, a_2, a_3, a_4\}),$$

and $u \subset r(u)$.

Proof. We start by rewriting the first condition

$$\begin{aligned} c &\in r(u), \\ c &\in (((u \downarrow)') \downarrow)', \\ c &\notin ((u \downarrow)') \downarrow, \\ (\{c\} \uparrow) \cap ((u \downarrow)') &= \emptyset, \\ (\{c\} \uparrow) &\subset (u \downarrow). \end{aligned}$$

We have a set $\{c\} \uparrow$ that is contained in a downward closed set $u \downarrow$. This is equivalent to all the maximal elements of $\{c\} \uparrow$ being in $u \downarrow$. The maximal elements of $\{c\} \uparrow$ must be among a_1, a_2, a_3, a_4 . Therefore, $c \in r(u)$ if and only if $(\{c\} \uparrow \cap \{a_1, a_2, a_3, a_4\}) \subset u \downarrow$. Any maximal element in a poset is in a downward closure $u \downarrow$ if and only if it is also in u .

Now notice that $(\{c\} \uparrow \cap \{a_1, a_2, a_3, a_4\}) \subset u \Leftrightarrow (\{c\} \uparrow \cap \{a_1, a_2, a_3, a_4\}) \subset (u \cap \{a_1, a_2, a_3, a_4\})$, so by the first part we get $r(u) = r(u \cap \{a_1, a_2, a_3, a_4\})$. Moreover, for $c \in u$, we have $\{c\} \uparrow \subset u \uparrow = u$, so $(\{c\} \uparrow \cap \{a_1, a_2, a_3, a_4\}) \subset (u \cap \{a_1, a_2, a_3, a_4\}) \subset u$ showing the inclusion. \square

Lemma 3.9. The sets $C_1 = \{u \in K : a_3 \in u\}, C_2 = \{u \in K : 2 \mid \#(u \cap \{a_1, a_2, a_3, a_4\})\}$ are images of clone candidates.

Proof. The mapping s restricted to the set $\{a_1, a_2, a_3, a_4\}$ is just the permutation (a_1, a_4) , and it is easy to see that both sets are closed under s . The closedness under r is implied by the previous lemma. If $u, v \in C_1$, then

$$\begin{aligned} a_3 &\in u \cap v, \\ a_3 &\notin u \div v, \\ a_3 &\notin (u \div v) \downarrow, \\ a_3 &\in ((u \div v) \downarrow)' = u \leftrightarrow v, \end{aligned}$$

so C_1 is closed under \leftrightarrow . On the other hand, if $u, v \in C_2$, then $u \cap v, u \cap v'$ have both odd or even number of elements from the set $\{a_1, a_2, a_3, a_4\}$. Similarly, $u \cap v', u' \cap v'$ has the same parity of elements from $\{a_1, a_2, a_3, a_4\}$. This leads to $(u \cap v) \cup (u' \cap v')$ having an even number of elements from $\{a_1, a_2, a_3, a_4\}$, so $u \leftrightarrow v \in C_2$. Finally, all members of C_1 contain a_3 and $\emptyset \in C_2$. This shows that the third property from the definition of clone candidates is also satisfied. \square

Lemma 3.10. The sets $C_3 = \{u \in K : 2 \mid \#(u \cap \{a_3, b_4, b_5, b_6\})\}, C_4 = \{u \in K : a_3 \in u, 2 \mid \#(u \cap \{b_1, b_3\}), 2 \mid \#(u \cap \{b_2, b_7\})\}$ are images of clone candidates.

Proof. The set C_3 is closed under s because $s|_{\{a_3, b_4, b_5, b_6\}}$ is the permutation (b_4, b_6) . On the other hand $s|_{\{a_3, b_1, b_2, b_3, b_7\}}$ is $(b_1, b_2)(b_3, b_7)$, so C_4 is also closed under s . By

Lemma 3.8, for any $i \in \{1, \dots, 11\}$, $b_i \in r(u)$ if and only if $b_i \uparrow \cap \{a_1, a_2, a_3, a_4\} \subset u$. This implies that if $a_3 \in u \in K$, then $\{a_3, b_4, b_5, b_6\} \subset r(u)$. Similarly, if $a_3 \notin u \in K$, then $\{a_3, b_4, b_5, b_6\} \cap r(u) = \emptyset$, so C_3 is closed under r . For C_4 , by a similar reasoning, we obtain that $r(u) \cap \{b_1, b_3\}$ is two-element or empty. These two cases can be recognized by checking whether $a_1 \in r(u)$. Similarly, $r(u) \cap \{b_2, b_7\}$ is two-element or empty depending on whether $a_4 \in r(u)$.

If $u, v \in C_3$ are such that a_3 belongs to exactly one of u, v , then $a_3 \in u \div v$, $a_3 \notin u \leftrightarrow v$, $\{a_3, b_4, b_5, b_6\} \cap (u \leftrightarrow v) = \emptyset$. If a_3 is not in u or in v , then $\{a_3, b_4, b_5, b_6\} \cap (u \div v) = \emptyset$, $\{a_3, b_4, b_5, b_6\} \subset u \leftrightarrow v$. If $a_3 \in u \cap v$ then intersections $u \cap \{b_4, b_5, b_6\}$, $v \cap \{b_4, b_5, b_6\}$ have an odd number of elements, so $(u \div v) \cap \{b_4, b_5, b_6\}$ has an even number of elements and $a_3 \notin u \div v$. In each case, $u \leftrightarrow v \in C_3$, so this set is closed under \leftrightarrow . For $u, v \in C_4$, we have $a_3 \notin u \div v$, $a_3 \in y \leftrightarrow v$ and $b_1 \in (u \div v) \Leftrightarrow b_3 \in (u \div v)$ so $b_1 \in (u \div v) \Downarrow \Leftrightarrow b_3 \in (u \div v) \Downarrow$ and $b_1 \in (u \leftrightarrow v) \Leftrightarrow b_3 \in (u \leftrightarrow v)$. An analogous argument works for the pair b_2, b_7 .

Finally, all members of C_4 contain a_3 , and $\emptyset \in C_3$. Therefore, the third property from the definition of clone candidate also holds. \square

Remark 3.11. The following inclusion holds: $C_4 \subsetneq C_1$, $\{M(x), M(y)\} \subset C_1 \cap C_2 \cap C_3 \cap C_4$.

We point out that we did not show that any of C_1, C_2, C_3, C_4 is actually an image of a clone, only that they are images of clone candidates.

Now we would like to show that the smallest possible image of a clone candidate is the intersection of these four distinguished subsets of K .

Lemma 3.12. The set $C_2 \cap C_3 \cap C_4$ is generated by $M(x), M(y)$ using \leftrightarrow and r .

Proof. Let $W = \{a_1, a_4, b_1, b_2, b_4, b_6, b_8, b_9, b_{10}, b_{11}\}$ and $f : C_2 \cap C_3 \cap C_4 \ni u \mapsto u \cap W$. Given the value of $f(u)$, one can compute whether the remaining elements a_2, a_3, b_3, b_5, b_7 belong to u from the definitions of C_2, C_3, C_4 . Namely, by definition of C_4 : $a_3 \in u$, $(b_3 \in u \Leftrightarrow b_1 \in f(u))$, $(b_7 \in u \Leftrightarrow b_2 \in f(u))$, by definition of C_3 : $b_5 \in u \Leftrightarrow 2|\#(f(u) \cap \{b_4, b_6\})$; and by definition of C_2 : $a_2 \in u \Leftrightarrow 2|\#(f(u) \cap \{b_1, b_3\})$. This shows that f is injective.

If we have any $v \in \text{Up}(W)$, then $v \cup \{a_3\}$ is upward closed in $\text{Cm}(\mathbf{F}_2)$. Notice that we can always add some of a_2, b_3, b_5, b_7 to $v \cup \{a_3\}$ in such a way that the obtained set is in $C_2 \cap C_3 \cap C_4$. This shows that for any $v \in \text{Up}(W)$ there is $u \in K$ with $f(u) = v$.

The pair $(C_2 \cap C_3 \cap C_4, \leftrightarrow)$ is an equivalential algebra with $A \leftrightarrow B = \text{Cm}(\mathbf{F}_2) \setminus ((A \div B) \downarrow)$; the pair $(\text{Up}(W), \leftrightarrow)$ is an equivalential algebra with the operation $A \leftrightarrow B = W \setminus ((A \div B) \downarrow)$. Now, we will show that f is not only a bijection but also an equivalential algebra isomorphism between these two structures.

Take any $u, v \in C_2 \cap C_3 \cap C_4$. From the definition $f(u \leftrightarrow v) = (\text{Cm}(\mathbf{F}_2) \setminus (u \div v) \downarrow) \cap W$, by elementary set theory this is equal to $(\text{Cm}(\mathbf{F}_2) \cap W) \setminus ((u \div v) \downarrow \cap W) = W \setminus ((u \div v) \downarrow \cap W)$. On the other hand, $f(u) \leftrightarrow f(v) = W \setminus ((u \cap W) \div (v \cap W)) \downarrow$. By elementary set theory, this is equal to $W \setminus ((u \div v) \cap W) \downarrow$. So we only need to show that if $A = u \div v \subset \text{Cm}(\mathbf{F}_2)$, then $(A \cap W) \downarrow = A \downarrow \cap W$.

We compute

$$\begin{aligned}
x \in (A \cap W) \downarrow &\Leftrightarrow \exists_{y \in A \cap W} y \geq x \\
&\Rightarrow (\exists_{y \in A} y \geq x) \& (\exists_{y \in W} y \geq x) \\
&\Leftrightarrow x \in (A \downarrow) \cap (W \downarrow) \\
&\Leftrightarrow x \in A \downarrow \cap W.
\end{aligned}$$

Hence, $((A \cap W) \downarrow) \subset (A \downarrow \cap W)$. To show the other inclusion take $x \in A \downarrow \cap W$. We have $x \in W$ and there exists $y \in A, y \geq x$. It follows that $y \in W \uparrow$, but $W \uparrow = W \cup \{a_3\}$. The case $y = a_3$ is not possible, because $a_3 \in u \cap v, a_3 \notin u \div v$. So we have $y \in W$, but then $y \in A \cap W, y \geq x$ so $x \in (A \cap W) \downarrow$. This shows that f is an (equivalential algebra) isomorphism.

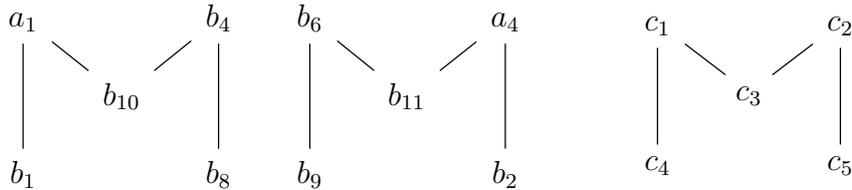


Figure 3.3: Left: The ordering of W . Right: The ordering of W^*

Looking at the structure of W , we see that it has two connected components

(Figure 3.3). This means that $(\text{Up}(W), \leftrightarrow)$ can be decomposed into the product of two copies of another equivalential algebra, $(\text{Up}(W^*), \leftrightarrow)$. The set $W^* = \{c_1, c_2, c_3, c_4, c_5\}$ is a poset ordered as depicted in Figure 3.3, and the equivalence operation is defined as usual. There are multiple ways of defining the isomorphism between these two structures; we want to select one of them. Define $f'' : W \mapsto \mathcal{P}(W^*) \times \mathcal{P}(W^*)$ by taking

$$\begin{aligned} f''(a_1) &= (\{c_1\}, \emptyset), f''(b_4) = (\{c_2\}, \emptyset), f''(b_{10}) = (\{c_3\}, \emptyset), \\ f''(b_3) &= (\{c_4\}, \emptyset), f''(b_8) = (\{c_5\}, \emptyset), \\ f''(a_4) &= (\emptyset, \{c_1\}), f''(b_6) = (\emptyset, \{c_2\}), f''(b_{11}) = (\emptyset, \{c_3\}), \\ f''(b_7) &= (\emptyset, \{c_4\}), f''(b_9) = (\emptyset, \{c_5\}). \end{aligned}$$

We define the isomorphism $f' : \text{Up}(W) \mapsto \text{Up}(W^*) \times \text{Up}(W^*)$ by $f'(u) = \bigcup_{d \in u} f''(d)$, where $f'(\emptyset) = (\emptyset, \emptyset)$. Proving that such a function is an equivalential algebra isomorphism follows from elementary set theory. One can for example show that f' preserves $\cap, \cup, ', \uparrow, \downarrow$ so it has to preserve \leftrightarrow .

So far, we have shown that $g = f' \circ f$ is an equivalential algebra isomorphism between $C_2 \cap C_3 \cap C_4$ and $\text{Up}(W^*) \times \text{Up}(W^*)$. Let us now consider the following six elements of $C_2 \cap C_3 \cap C_4$:

$$\begin{aligned} M(r(x)) &= \{a_3, a_4, b_2, b_4, b_5, b_6, b_7, b_8, b_9, b_{11}\} = \{a_1, a_2, b_1, b_3, b_{10}\}', \\ M(yxxy) &= \{a_1, a_2, a_3, a_4, b_2, b_4, b_5, b_6, b_7, b_9, b_{11}\} = \{b_1, b_3, b_8, b_{10}\}', \\ M(d(x)yy) &= \{a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_6, b_7, b_9, b_{11}\} = \{b_4, b_5, b_8, b_{10}\}', \\ M(r(y)) &= \{a_1, a_3, b_1, b_3, b_4, b_5, b_6, b_8, b_9, b_{10}\} = \{a_2, a_4, b_2, b_7, b_{11}\}', \\ M(xyyx) &= \{a_1, a_2, a_3, a_4, b_1, b_3, b_4, b_5, b_6, b_8, b_{10}\} = \{b_2, b_7, b_9, b_{11}\}', \\ M(d(y)xx) &= \{a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, b_7, b_8, b_{10}\} = \{b_5, b_6, b_9, b_{11}\}'. \end{aligned}$$

For each term t among the above six, $M(t)$ can either be computed directly, or one can just check Figure 1.3 for all subdirectly irreducible images of \mathbf{F}_2 , in which $t = 1$. If we project those six elements by g , we get

$$\begin{aligned}
g(M(r(x))) &= (\{c_2, c_5\}, W^*) \\
g(M(yxxy)) &= (\{c_1, c_2\}, W^*), \\
g(M(d(x)yy)) &= (\{c_1, c_4\}, W^*), \\
g(M(r(y))) &= (W^*, \{c_2, c_5\}), \\
g(M(xy yx)) &= (W^*, \{c_1, c_2\}), \\
g(M(d(y)xx)) &= (W^*, \{c_1, c_4\}).
\end{aligned}$$

In $\text{Up}(W^*)$, one can calculate that

$$\begin{aligned}
\emptyset &= \{c_1, c_4\} \leftrightarrow \{c_2, c_5\}, \\
W^* &= \emptyset \leftrightarrow \emptyset, \\
\{c_1\} &= \{c_1, c_2\} \leftrightarrow \{c_1, c_4\}, \\
\{c_2\} &= \{c_1, c_2\} \leftrightarrow \{c_2, c_5\}, \\
\{c_1, c_2, c_3, c_4\} &= \{c_2, c_5\} \leftrightarrow \{c_2\}, \\
\{c_1, c_2, c_3, c_5\} &= \{c_1, c_4\} \leftrightarrow \{c_1\}, \\
\{c_1, c_2, c_5\} &= \{c_1, c_2, c_3, c_4\} \leftrightarrow \{c_1, c_2\}, \\
\{c_1, c_2, c_4\} &= \{c_1, c_2, c_3, c_5\} \leftrightarrow \{c_1, c_2\}, \\
\{c_1, c_2, c_3\} &= \{c_1, c_2, c_4\} \leftrightarrow \{c_1, c_2, c_5\}, \\
\{c_1, c_2, c_4, c_5\} &= \{c_1, c_2, c_3\} \leftrightarrow \{c_1, c_2\},
\end{aligned}$$

hence the triple $\{c_1, c_2\}, \{c_1, c_4\}, \{c_2, c_5\}$ generates equivalential algebra $(\text{Up}(W^*), \leftrightarrow)$. This means that the sextuple

$$\begin{aligned}
&(\{c_2, c_5\}, W^*), (\{c_1, c_2\}, W^*), (\{c_1, c_4\}, W^*), \\
&(W^*, \{c_2, c_5\}), (W^*, \{c_1, c_2\}), (W^*, \{c_1, c_4\})
\end{aligned}$$

generates the equivalential algebra with the universe $\text{Up}(W^*) \times \text{Up}(W^*)$. Because g is an isomorphism, the sextuple

$$M(r(x)), M(yxxy), M(d(x)yy), M(r(y)), M(xy yx), M(d(y)xx)$$

must generate the algebra $(C_2 \cap C_3 \cap C_4, \leftrightarrow)$. Because all six of those terms can be easily written using equivalence and regularization, we conclude that $(C_2 \cap C_3 \cap C_4, \leftrightarrow, r)$ is generated by the pair $M(x), M(y)$.

□

Lemma 3.13. Let C be a clone candidate. The following holds

$$\begin{aligned} M(C) \not\subset C_1 &\Leftrightarrow \neg x \in C, \\ M(C) \not\subset C_2 &\Leftrightarrow x \hat{\wedge} y \in C, \\ x \hat{\wedge} y \in C &\Rightarrow C_3 \cap C_4 \subset M(C). \end{aligned}$$

Proof. If $M(C) \not\subset C_1$, then from the definition there exists $t \in C$ such that $a_3 \notin M(t)$. By the definition of candidate clone we have $\emptyset \in M(C)$. Hence, $\neg x = 0 \cdot x = M^{-1}(\emptyset \leftrightarrow M(x))$ must be an element of C . On the other hand, if $\neg x \in C$, then $M(\neg x) \in M(C)$ and $M(\neg x) \notin C_1$.

If $M(C) \not\subset C_2$, then there exists $t \in C$ such that $2 \nmid \#M(t) \cap \{a_1, a_2, a_3, a_4\}$. We have two possibilities depending on whether $a_3 \in M(t)$. If $a_3 \in M(t)$, then take the element $u = (\{a_1, a_2, a_4\} \setminus M(t)) \cup \{a_3, b_5\}$. The set u is an element of $C_2 \cap C_3 \cap C_4$, which is the image of the minimal clone, so it must also be in $M(C)$. Therefore, the element $M(t) \leftrightarrow u$ must also be contained in the clone $M(C)$, and we have

$$\begin{aligned} (M(t) \leftrightarrow u) \cap \{a_1, a_2, a_3, a_4\} &= \\ &= (M(t) \div u)' \cap \{a_1, a_2, a_3, a_4\} \\ &= \{a_1, a_2, a_3, a_4\} \setminus (M(t) \div u) \\ &= \{a_1, a_2, a_3, a_4\} \setminus ((M(t) \cap \{a_1, a_2, a_3, a_4\}) \div (u \cap \{a_1, a_2, a_3, a_4\})) \\ &= \{a_1, a_2, a_3, a_4\} \setminus \{a_1, a_2, a_4\} \\ &= \{a_3\}. \end{aligned}$$

This means, by Lemma 3.8, that $r(M(t) \leftrightarrow u) = r(\{a_3\})$ is also an element of C .

On the other hand, if $a_3 \notin M(t)$, then, by the previous point, $\neg x \in C$. As $M(\neg x) = \{a_1, a_2, b_3\}$, the element $M(t \cdot (\neg x))$ is not in C_2 but it contains a_3 and we reduced the situation to the previous case.

Hence, $M(C) \not\subset C_2$ implies $r(\{a_3\}) \in M(C)$, and it remains to notice that $r(\{a_3\}) = r(M(x \wedge y)) = M(r(x \wedge y)) = M(x \widehat{\wedge} y)$. We also have that if $x \widehat{\wedge} y \in C$, then $M(x \widehat{\wedge} y) \in M(C)$, $M(x \widehat{\wedge} y) \notin C_2$.

Assume that $r(\{a_3\}) \in M(C)$, and take any $v \in C_3 \cap C_4$. We have $r(v \leftrightarrow r(v)) = r(v) \leftrightarrow r(v) = 1$, so by Lemma 3.8, $\{a_1, a_2, a_3, a_4\} \subset (v \leftrightarrow r(v))$. By definition this means that $(v \leftrightarrow r(v)) \in C_2$, and as $C_3 \cap C_4$ is closed on operations, $(v \leftrightarrow r(v)) \in C_2 \cap C_3 \cap C_4 \subset M(C)$.

Because the set $C_3 \cap C_4$ is closed under operations, $r(v) \in C_3 \cap C_4$. Assume now that $r(v) \notin C_2$. Because $r(v) \in C_4$ we have $a_3 \in r(v)$. By Lemma 3.8, a regular element is determined by its intersection with $\{a_1, a_2, a_3, a_4\}$, so $r(v)$ is one of the four sets:

$$\begin{aligned} r(v) &= r(\{a_3\}) = M(x \widehat{\wedge} y) \quad \text{or} \\ r(v) &= r(\{a_1, a_2, a_3\}) = M(x \widehat{\wedge} y) \leftrightarrow r(M(x)) \quad \text{or} \\ r(v) &= r(\{a_1, a_3, a_4\}) = (M(x \widehat{\wedge} y) \leftrightarrow r(M(x))) \leftrightarrow r(M(y)) \quad \text{or} \\ r(v) &= r(\{a_2, a_3, a_4\}) = M(x \widehat{\wedge} y) \leftrightarrow r(M(y)). \end{aligned}$$

In each case we have $r(v) \in M(C)$. If $r(v) \in C_2$, then $r(v) \in C_2 \cap C_3 \cap C_4$, and by the minimality of that set, $r(v) \in M(C)$. Finally, $v = (v \leftrightarrow r(v)) \leftrightarrow r(v) \in M(C)$. \square

Lemma 3.14. If C is a clone candidate, then

$$M(C) \not\subset C_3 \Leftrightarrow x \square y \in C.$$

Moreover, if the above equivalent conditions are true, then for any $v \in K$ such that $a_3 \in v$, we have $(v \in M(C)) \Leftrightarrow ((v \div \{b_5\}) \in M(C))$.

Proof. If $M(C) \not\subset C_3$, then there exists $t \in C$ such that $2 \nmid \#(M(t) \cap \{a_3, b_4, b_5, b_6\})$. We will construct a series of elements with the goal of showing that $M(x \square y) = \{a_1, a_2, a_3, a_4\} \in M(C)$.

Take $u_1 = M(t) \leftrightarrow r(M(t))$. Because $2 \nmid \#(M(t) \cap \{a_3, b_4, b_5, b_6\})$ and $M(t)$ is upward closed, we must have $a_3 \in M(t)$, by Lemma 3.8 $\{a_3, b_4, b_5, b_6\} \subset r(M(t))$. It follows that $M(t) \div r(M(t))$ does not contain a_3 and it contains an odd number of b_4, b_5, b_6 . The same applies to $(M(t) \div r(M(t))) \downarrow$, so $u_1 \notin C_3$. Define $u_2 =$

$u_1 \leftrightarrow M(xy) = u_1 \leftrightarrow \{a_2, a_3, b_5\}$ and $u_3 = u_2 \leftrightarrow M(xy)$, by a similar argument they satisfy $u_2, u_3 \notin C_3$. However, $r(u_1) = r(M(t)) \leftrightarrow r(r(M(t)) = 1$, so $r(u_2) = r(u_1) \leftrightarrow r(M(xy)) = r(\{a_2, a_3\})$ and $r(u_3) = 1$. This lets us conclude that $a_1, a_4 \notin u_2$, so $b_1, b_2, b_3, b_7 \notin u_2$ and $u_2 \in C_4$. As $M(xy) \in C_2 \cap C_3 \cap C_4 \subset M(C)$, we get $u_1, u_2, u_3 \in M(C)$ and $u_3 \in C_2 \cap C_4, u_3 \notin C_3$.

Take $u_4 = (\{b_1, b_2, b_3, b_4, b_5, b_6, b_7\} \setminus u_3) \cup \{a_1, a_2, a_3, a_4\}$. By looking at the structure of $\text{Cm}(\mathbf{F}_2)$ this set is upwards closed. Because $u_3 \in C_4$, so does u_4 . We also have that $u_4 \in C_2$ because it contains all maximal elements, and $u_4 \in C_3$ because $u_3 \cap \{b_4, b_5, b_6\}$ contains 0 or 2 elements. Hence $u_4 \in C_2 \cap C_3 \cap C_4 \subset M(C)$. Notice that $\{b_1, b_2, b_3, b_4, b_5, b_6, b_7\} \subset (u_4 \div u_3)$ and $\{a_1, a_2, a_3, a_4\} \subset u_3 \cap u_4$. Therefore,

$$\begin{aligned} u_3 \leftrightarrow u_4 &= ((u_3 \div u_4) \downarrow)' \\ &= \{b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}, b_{11}\}' \\ &= \{a_1, a_2, a_3, a_4\} \\ &= M(x \sqcap y) \end{aligned}$$

is an element of $M(C)$. On the other hand, if $x \sqcap y \in C$, then $M(x \sqcap y) \in M(C)$, $M(x \sqcap y) \notin C_3$.

Assume that $x \sqcap y \in C$. We also have $\{a_1, a_2, a_3, a_4, b_5\} \in C_2 \cap C_3 \cap C_4 \subset M(C)$. Because a clone candidate is closed under equivalence we have

$$\begin{aligned} M(x \sqcap y) \leftrightarrow \{a_1, a_2, a_3, a_4, b_5\} &= ((\{a_1, a_2, a_3, a_4\} \div \{a_1, a_2, a_3, a_4, b_5\}) \downarrow)' \\ &= (\{b_5\} \downarrow)' \\ &= \text{Cm}(\mathbf{F}_2) \setminus \{b_5\} \\ &= (\text{Cm}(\mathbf{F}_2) \div \{b_5\}) \in M(C). \end{aligned}$$

Now, remember that \div is associative. Therefore, for any $v \in M(C)$ such that $a_3 \in v$, we have

$$\begin{aligned} v \leftrightarrow (\text{Cm}(\mathbf{F}_2) \div \{b_5\}) &= ((v \div \text{Cm}(\mathbf{F}_2) \div \{b_5\}) \downarrow)' \\ &= (((v \div \{b_5\})') \downarrow)'. \end{aligned}$$

Because $a_3 \in v$, the set $v \div \{b_5\}$ is upward closed, so $(v \div \{b_5\})'$ is downward closed. We get $v \leftrightarrow (\text{Cm}(\mathbf{F}_2) \div \{b_5\}) = v \div \{b_5\}$, and by closedness under equivalence $v \div \{b_5\} \in M(C)$.

This shows one implication $((v \in M(C)) \Rightarrow ((v \div \{b_5\}) \in M(C)))$. To show the other implication, it is enough to notice that $v = v \div \{b_5\} \div \{b_5\}$. \square

Lemma 3.15. If C is a clone candidate, then

$$M(C) \not\subset C_4 \Leftrightarrow x - y \in C.$$

Moreover, if the above equivalent conditions are true, then for any $v \in K, a_4 \in v$ we have $(v \in M(C)) \Leftrightarrow ((v \div \{b_7\}) \in M(C))$. For any $u \in K, a_1 \in u$, we have $(u \in M(C)) \Leftrightarrow ((u \div \{b_3\}) \in M(C))$.

Proof. If $M(C) \not\subset C_1$, then $\neg x \in C$ such that $y - x = y(\neg x)(\neg x) \in C$, and because C is closed under symmetry, $x - y \in C$. On the other hand, if $M(C) \subset C_1$ and $M(C) \not\subset C_4$, then there exists $t \in C$ such that $M(t) \not\subset C_4, M(t) \in C_1$. By closedness under symmetry, we can assume $\#(M(t) \cap \{b_2, b_7\}) = 1$. We would like to show that $M(x - y) = (\{a_1, a_2, b_7\} \downarrow)'$ is an element of $M(C)$.

We will construct a series of elements in the clone $M(C)$ starting from t , such that each of them is not in C_4 . Moreover, on every step we will get some extra details about the term. To simplify some calculations, we define two helper sets:

$$U = \{u \in K : u \cap \{a_3, a_4, b_2, b_7\} = \{a_3, a_4, b_2\} \text{ or } u \cap \{a_3, a_4, b_2, b_7\} = \{a_3, a_4, b_7\}\},$$

$$V = \{v \in K : v \cap \{a_3, a_4, b_2, b_7\} = \{a_3, a_4\} \text{ or } v \cap \{a_3, a_4, b_2, b_7\} = \{a_3, a_4, b_2, b_7\}\}.$$

They were chosen in such a manner that for any $u \in U, v \in V$ the set $(u \div v) \cap \{a_3, a_4, b_2, b_7\}$ is equal to $\{b_2\}$ or $\{b_7\}$. Because $\{a_3, a_4, b_2, b_7\}$ is upward closed, we obtain $(u \div v) \downarrow \cap \{a_3, a_4, b_2, b_7\} = (u \div v) \cap \{a_3, a_4, b_2, b_7\}$, so $u \leftrightarrow v \in U$. This is a useful observation because $M(t) \in U$ and $U \cap C_4 = \emptyset$. Therefore, as long as all the constructed elements are in U , they cannot be in C_4 . We also observe that $M(r(x)) = \{a_3, a_4, b_2, b_4, b_5, b_6, b_7, b_8, b_9, b_{11}\} \in V$. For easier tracking of the construction, we present the following table. Each row describes an image of a term, sets to which it belongs, and some information about its elements. The underlining indicates the property that was the goal of a given step of construction.

element	sets it belongs to	what is the size of the intersection with:						
		$\{a_3\}$	$\{b_4, b_5, b_6\}$	$\{a_4\}$	$\{b_2, b_7\}$	$\{a_1\}$	$\{b_1, b_3\}$	$\{a_2\}$
$M(t)$	U	1	?	1	1	?	?	?
$M(r(x))$	$V, C_2 \cap C_3 \cap C_4$	1	3	1	2	0	0	0
$\{a_2\}'$	$V, C_3 \cap C_4$	1	3	1	2	1	2	0
w_1	U	1	?	1	1	<u>0</u>	0	?
w_2	U	1	?	1	1	1	2	?
$s(w_2)$	V	1	?	1	2	1	1	?
w_3	U, C_3	1	<u>odd</u>	1	1	1	1	?
w_4	U, C_3	1	odd	1	1	<u>0</u>	0	?
w_5	U, C_3	1	odd	1	1	0	0	<u>0</u>
w_6	$V, C_2 \cap C_3 \cap C_4$	1	odd	1	even	1	0	1
w_7	U, C_3	1	odd	1	1	0	0	0

We start by taking $w_1 = M(t)$ or $w_1 = M(t) \leftrightarrow M(r(x))$ such that $a_1 \notin w_1$. Because $M(t) \in U$ and $M(r(x)) \in V$ we have $w_1 \in U$. Next, we define $w_2 = w_1 \leftrightarrow M(r(x))$, which is an element of U . Because $w_1 \cap \{a_1, b_1, b_3\} = M(r(x)) \cap \{a_1, b_1, b_3\} = \emptyset$, we have $\{a_1, b_1, b_3\} \subset w_2$. This implies $\{a_4, b_2, b_7\} = s(a_1, b_1, b_3) \subset s(w_2)$, so $s(w_2) \in V$ and we take $w_3 = w_2 \leftrightarrow s(w_2) \in U$. By symmetry, the set $w_2 \div s(w_2)$ does not contain a_3, b_5 and contains both or none of b_4, b_6 . The same applies to the set $(w_2 \div s(w_2)) \downarrow$. This ensures $w_3 \in C_3$. Next, we take $w_4 = w_3 \leftrightarrow M(r(x))$, and we have $w_4 \in C_3 \cap U$. Moreover, $a_1 \notin w_4$ and $a_3, a_4 \in w_4$. Now, we check whether $w_4 \in C_2$, which is equivalent to $a_2 \notin w_4$. If $w_4 \in C_2$, then take $w_5 = w_4$. Otherwise, by Lemma 3.13, $C_3 \cap C_4 \subset M(C)$, $\text{Cm}(\mathbf{F}_2) \setminus \{a_2\} \in M(C)$, and take $w_5 = w_4 \leftrightarrow (\{a_2\}') \in M(C)$. Because $\{a_2\}' \in V$ we have $w_5 \in U$ and it is easy to verify that $w_5 \in C_2$.

We have an element $w_5 \in U \cap C_2 \cap C_3$, which must be in the clone $M(C)$ and $w_5 \cap \{a_1, a_2, b_1, b_3\} = \emptyset$. The set $w_5 \in U$ contains exactly one of b_2, b_7 . Take $w_6 = w_5 \cup \{a_1, a_2, b_7\}$ if $b_2 \in w_5$ or $w_6 = (w_5 \cup \{a_1, a_2\}) \setminus \{b_7\}$ otherwise. Both of these lead to $w_6 \in C_2 \cap C_3 \cap C_4 \cap V$ and we have $w_6 \div w_5 = \{a_1, a_2, b_7\}$. After this lengthy construction, we can just take $w_7 = w_6 \leftrightarrow w_5 = (\{a_1, a_2, b_7\} \downarrow)'$, so $M(x - y) \in M(C)$ as intended.

On the other hand, if $x - y \in C$, then $M(x - y) \in M(C)$ and $M(x - y) \notin C_4$.

Assume $M(x - y) \in M(C)$. It is easy to verify that $(\{a_1, a_2\} \downarrow)' \in C_2 \cap C_3 \cap C_4$.

We have

$$\begin{aligned}
M(x - y) \leftrightarrow (\{a_1, a_2\} \downarrow)' &= (((\{a_1, a_2, b_7\} \downarrow)' \div (\{a_1, a_2\} \downarrow)') \downarrow)' \\
&= ((\{a_1, a_2, b_7\} \downarrow \div \{a_1, a_2\} \downarrow) \downarrow)' \\
&= (\{b_7\} \downarrow)' \\
&= \text{Cm}(\mathbf{F}_2) \div \{b_7\},
\end{aligned}$$

therefore $\{b_7\}'$ is an element of $M(C)$. If $v \in M(C)$ and $a_4 \in v$, then $v \div \{b_7\}$ is upward closed, so its complement is downward closed. Therefore,

$$\begin{aligned}
v \leftrightarrow (\text{Cm}(\mathbf{F}_2) \div \{b_7\}) &= ((v \div \text{Cm}(\mathbf{F}_2) \div \{b_7\}) \downarrow)' \\
&= (((v \div \{b_7\})') \downarrow)' \\
&= v \div \{b_7\}
\end{aligned}$$

and $v \div \{b_7\}$ is also an element of $M(C)$. Now, it remains to notice that the function $v \mapsto (v \div \{b_7\})$ is an involution. By analogous reasoning, we get that for any $a_1 \in u \in K$ the equivalence $(u \in M(C)) \Leftrightarrow ((u \div \{b_3\}) \in M(C))$ (or use the fact that $M(C)$ is closed under the symmetry mapping s). \square

Theorem 3.16. *If C is a clone candidate then $M(C) = K$ or $M(C)$ is an intersection of some of C_1, C_2, C_3, C_4 .*

Proof. In equivalential algebras with regularization, any element can be represented by $u = r(u)d(u)$. This means that the equality $M(C) = \{u \leftrightarrow v : u, v \in M(C), u = r(u), r(v) = 1\}$ holds and we only need to describe the dense and regular elements in $M(C)$. We will first show that there are four possibilities for $r(M(C))$ and four for $d(M(C))$.

If $u \in K$ is regular, then from Lemma 3.8 it can be uniquely identified by $u \cap \{a_1, a_2, a_3, a_4\}$, so there are sixteen regular elements in K , which correspond to binary connectives in Boolean logic. Four of those are always in $M(C)$: $M(x)$, $M(y)$, $M(xy)$, $M(1)$. If $M(C) \not\subseteq C_1$, then $M(\neg x) \in M(C)$ by Lemma 3.13. Because the clone candidate is closed under equivalence, $M(\neg y)$, $M(\neg xy)$, $M(\neg 1) \in M(C)$. On the other hand, if $M(C) \not\subseteq C_2$, then $M(x \hat{\wedge} y) \in M(C)$ and, by the clone candidate

being closed under equivalence, $M(x(x\hat{\wedge}y)), M(y(x\hat{\wedge}y)), M(xy(x\hat{\wedge}y)) \in M(C)$. Finally, if $M(C) \not\subset C_1 \cup C_2$, then $M(\neg x), M(x\hat{\wedge}y) \in M(C)$ and by closedness under equivalence all sixteen possible regular elements are in $M(C)$. This means that the subset of regular elements of $M(C)$ is equal to one of $r(C_1 \cap C_2), r(C_1), r(C_2), r(K)$.

Now we look at dense elements. The set $v \in K$ is dense if and only if $r(v) = 1$. By Lemma 3.8, this is equivalent to $\{a_1, a_2, a_3, a_4\} \subset v$. It follows that $d(K) \subset C_1 \cap C_2$. Similarly to the regular case, we can divide dense elements into four sets: those that are in $C_3 \cap C_4$, which are always contained in $M(C)$ (by Lemma 3.12); those that are in $C_3 \setminus C_4$; those that are in $C_4 \setminus C_3$; and those that are in $K \setminus (C_3 \cup C_4)$.

If $M(C) \not\subset C_3$, then by Lemma 3.14 the set $d(M(C))$ must be closed under the involution $f_1 : v \mapsto (v \div \{b_5\})$. Notice that f_1 maps elements of $d(C_4 \setminus C_3)$ to elements of $d(C_4 \cap C_3)$ and vice versa. Hence $d(C_4 \setminus C_3) \subset M(C)$.

Similarly, if $M(C) \not\subset C_4$, then we can use Lemma 3.15. The set $d(M(C))$ must be closed under two involutions $f_2 : v \mapsto (v \div \{b_3\}), f_3 : v \mapsto (v \div \{b_7\})$. If $v \in d(C_3 \setminus C_4)$, then one of the images $f_2(v), f_3(v), (f_2 \circ f_3)(v)$ belongs to $C_2 \cap C_3 \cap C_4$. Therefore, $d(C_3 \setminus C_4) \subset M(C)$.

In the last case, if $M(C)$ contains an element that is neither in C_3 nor C_4 , then $d(M(C))$ is closed under the functions f_1, f_2, f_3 and their compositions. Note that these three functions are commuting involutions. Let α be an equivalence relation on $d(K)$ given by

$$u\alpha v \Leftrightarrow u \setminus \{b_3, b_5, b_7\} = v \setminus \{b_3, b_5, b_7\}.$$

Relation α divides $d(K)$ into octuples that are closed under f_1, f_2, f_3 . Therefore, each octuple is contained in $d(M(C))$ or has no common element with it. But in each octuple there is exactly one element that is also in $C_2 \cap C_3 \cap C_4$. By Lemma 3.12, $C_2 \cap C_3 \cap C_4 \subset M(C)$, so each α -coset has an element in $d(M(C))$. Therefore, every coset is in $d(M(C))$, and $d(K) \subset d(M(C))$. To sum up, $d(M(C))$ is one of the sets $d(C_3 \cap C_4), d(C_3), d(C_4), d(K)$.

As mentioned in the beginning, $M(C) = r(M(C))d(M(C))$, so there are at most

sixteen possibilities for $M(C)$:

$$\begin{aligned}
& r(C_1 \cap C_2)d(C_3 \cap C_4), r(C_1 \cap C_2)d(C_3), r(C_1 \cap C_2)d(C_4), r(C_1 \cap C_2)d(K), \\
& r(C_1)d(C_3 \cap C_4), r(C_1)d(C_3), r(C_1)d(C_4), r(C_1)d(K), \\
& r(C_2)d(C_3 \cap C_4), r(C_2)d(C_3), r(C_2)d(C_4), r(C_2)d(K), \\
& r(K)d(C_3 \cap C_4), r(K)d(C_3), r(K)d(C_4), r(K)d(K).
\end{aligned}$$

We have that if $r(M(C)) = r(K)$ or $r(M(C)) = r(C_2)$, then $M(C) \not\subseteq C_1$ and, by Lemma 3.13, $M(\neg x) \in M(C)$. Therefore, $M(x - y) \in M(C)$ and $M((x - y)r(x)) = \{b_7\}'$ is a dense element that is not in C_4 . Hence, we can remove the following four possibilities: $r(C_2)d(C_3 \cap C_4), r(C_2)d(C_4), r(K)d(C_3 \cap C_4), r(K)d(C_4)$.

By direct check, $r(C_3) = r(K), r(C_4) = r(C_1)$ and $d(C_1) = d(C_2) = d(K)$. If we consider $C_2 \cap C_3 \cap C_4$, then $r(C_2 \cap C_3 \cap C_4) = r(C_1 \cap C_2)$ and $d(C_2 \cap C_3 \cap C_4) = d(C_3 \cap C_4)$, so

$$C_2 \cap C_3 \cap C_4 = r(C_1 \cap C_2)d(C_3 \cap C_4).$$

Similarly, we rewrite the remaining eleven cases:

$$\begin{aligned}
C_1 \cap C_2 \cap C_3 &= r(C_1 \cap C_2)d(C_3), \\
C_2 \cap C_4 &= r(C_1 \cap C_2)d(C_4), \\
C_1 \cap C_2 &= r(C_1 \cap C_2)d(K), \\
C_3 \cap C_4 &= r(C_1)d(C_3 \cap C_4), \\
C_3 \cap C_1 &= r(C_1)d(C_3), \\
C_4 &= r(C_1)d(C_4), \\
C_1 &= r(C_1)d(K), \\
C_2 \cap C_4 &= r(C_2)d(C_3), \\
C_2 &= r(C_2)d(K), \\
C_3 &= r(K)d(C_3), \\
K &= r(K)d(K).
\end{aligned}$$

This shows that each possibility is equal to K , or is an intersection of some of C_1, C_2, C_3, C_4 . □

So far, we have considered the clones that contain regularization. Now we will show that, without regularization, it is not possible to have a mixed type. If we have any term t that is in the clone generated by $\{\leftrightarrow, \neg\}$, then t can be written as a polynomial using \leftrightarrow and constant 0. It follows that (\leftrightarrow, t) -reducts have the same polynomial clone as (\leftrightarrow) -reducts, and hence their type is **2**. We are working with congruence permutable algebras, so by Corollary 1.73, also the subreducts are of type **2**. Therefore, there remains only one possible situation to consider.

Lemma 3.17. If $t \in \mathbf{F}_2$ is such that $r(x)$ is not in the clone generated by $\{\leftrightarrow, t\}$ and t is not in the clone generated $\{\leftrightarrow, \neg\}$, then $a_3 \in M(t)$ and

$$M(t) \cap \{a_1, a_2, a_4, b_4, b_5, b_6\} \in \{\emptyset, \{a_2, a_4, b_5, b_6\}, \{a_1, a_4, b_4, b_6\}, \{a_1, a_2, b_4, b_5\}\}.$$

Proof. The clone generated by $\{\leftrightarrow, \neg\}$ is a set $C \subset F_2$. It must, of course, satisfy the conditions for being a clone candidate. By Lemma 3.13, $M(C) \not\subset C_1$ because $\neg x \in C$. By the previous theorem, $M(C)$ must be one of $K, C_2, C_3, C_2 \cap C_3$. In any way, $C_2 \cap C_3 \subset M(C)$.

By Corollary 3.6 if $r(x)$ is not in the clone generated by $\{\leftrightarrow, t\}$, then $a_3 \in M(t)$ and $2|\#\(\{a_2, b_5\} \cap M(t)\), $2|\#\(\{a_1, b_4\} \cap M(t)\), $2|\#\(\{a_4, b_6\} \cap M(t)\)$. This restricts $M(t) \cap \{a_1, a_2, a_4, b_4, b_5, b_6\}$ to eight possibilities. But if $M(t) \not\subset M(C)$, then $t \notin (C_2 \cap C_3)$ so $2|\#\(\{a_1, a_2, a_4\} \cap M(t)\)$ or $2|\#\(\{b_4, b_5, b_6\} \cap M(t)\)$. $\square$$$

In the above proof, we used the fact that $C_2 \cap C_3$ is contained in the image of the clone C generated by $\{\leftrightarrow, \neg\}$. In fact $M(C) = C_2 \cap C_3$. If $M(C)$ would contain any element not from $C_2 \cap C_3$ then by Lemma 3.13 or Lemma 3.14 we would have $x \widehat{\wedge} y \in C$ or $x \sqcap y \in C$. However, both of those operations introduce a mixed type, and (\leftrightarrow, \neg) -subreducts are of type **2**.

By direct computation, we get that $17 * 17 = 289$ terms satisfy the above lemma. This is exactly half of those terms t , that the clone generated by $\{\leftrightarrow, t\}$ does not contain regularization.

Lemma 3.18. If $t \in \mathbf{F}_2$ satisfies the requirements of Lemma 3.17, then one of $t(x, y), t(x, y)x, t(x, y)y, t(x, y)xy$ is equal to $x \wedge y$.

Proof. If t satisfies the conditions of Lemma 3.17, then the conditions are also true for the four terms $t(x, y), t(x, y)x, t(x, y)y, t(x, y)xy$. In particular, each of the sets $M(t), M(tx), M(ty), M(txy)$ has an even-sized intersection with $\{a_1, a_2, a_4\}$, and contains a_3 .

On the other hand, none of $r(M(t)), r(M(tx)), r(M(ty)), r(M(txy))$ can be equal to each other, because $1 \neq r(M(x)) \neq r(M(y)) \neq 1$ and M is an isomorphism. By Lemma 3.8, each of the sets $M(t), M(tx), M(ty), M(txy)$ must give a different result when intersected with $\{a_1, a_2, a_3, a_4\}$.

Combining both facts together, one of $M(t), M(tx), M(ty), M(txy)$ intersected with $\{a_1, a_2, a_3, a_4\}$ is equal to $\{a_3\}$. But then, by Lemma 3.17, this set must have an empty intersection with $\{a_1, a_2, a_4, b_4, b_5, b_6\}$. If we look back at the structure of $\text{Cm}(\mathbf{F}_2)$, the only upward closed set that satisfies this condition is $\{a_3\}$. From Lemma 3.2, we get $x \wedge y = M^{-1}(\{a_3\})$. \square

The above lemma implies that the clone generated by $\{\leftrightarrow, t\}$ contains \wedge . Therefore, any (\leftrightarrow, t) -subreduct \mathbf{A} has a Brouwerian semilattice reduct \mathbf{B} . We have $\text{Con}(\mathbf{A}) \leq \text{Con}(\mathbf{B})$, and \mathbf{B} is congruence distributive. By Corollary 1.73, \mathbf{A} is of type **3**. To sum up the results so far:

Theorem 3.19. *If $t \in \mathbf{F}_2$, and $r(x)$ does not belong to the clone generated by $\{\leftrightarrow, t\}$, then \wedge is in the clone generated by $\{\leftrightarrow, t\}$ or t is in the clone generated by $\{\leftrightarrow, \neg\}$. In the first case, the subreducts are always of type **3**. In the second case, they are of type **2**.*

This means that \mathcal{G} -subreducts can be of mixed type only if $r \in \mathcal{G}$. Moreover, if we take the set $G_{\mathcal{G}} \subset \mathbf{F}_2$ that contains all the elements generated from x, y using terms from \mathcal{G} , then $G_{\mathcal{G}}$ is a clone candidate. We have identified that there are only 12 clone candidates. Lemmas 3.13, 3.14, 3.15 describe when certain elements belong to $G_{\mathcal{G}}$. If $\widehat{\wedge}, \sqcap \in G_{\mathcal{G}}$, then also $\wedge \in G_{\mathcal{G}}$, and \mathcal{G} -subreducts are of type **3**. On the other hand, if $\widehat{\wedge}, \sqcap \notin G_{\mathcal{G}}$, then $M(G_{\mathcal{G}}) \subset C_2 \cap C_3$. It follows that $G_{\mathcal{G}}$ is a subset of the clone generated by $\{\leftrightarrow, \neg\}$ and \mathcal{G} -reducts are of type **2**.

Therefore, there are only six clone candidates that can yield mixed type algebra. Their respective images by M are $C_4 \cap C_3, C_1 \cap C_3, C_3, C_4 \cap C_2, C_1 \cap C_2, C_2$. So far we

only verified that the six listed sets satisfy the necessary conditions to be an image of a clone, but this means that there are at most six classes of subreducts that we are interested in. Now we can just check that there are at least six different classes of subreducts. We take the following:

- EARS, $(\leftrightarrow, \widehat{\wedge})$ -subreducts (corresponding to $C_4 \cap C_2$);
- $(\leftrightarrow, \widehat{\wedge}, -)$ -subreducts (corresponding to $C_1 \cap C_2$);
- $(\leftrightarrow, \widehat{\wedge}, \neg)$ -subreducts (corresponding to C_2);
- EADS, $(\leftrightarrow, \sqcap)$ -subreducts (corresponding to $C_4 \cap C_3$);
- $(\leftrightarrow, \sqcap, -)$ -subreducts (corresponding to $C_1 \cap C_3$);
- $(\leftrightarrow, \sqcap, \neg)$ -subreducts (corresponding to C_3).

To check that each of those classes contain a mixed type algebra, it is enough to take a respective reduct of the three-element Heyting chain. The commutator operation in these reducts behaves differently in the first three than in the last three. To show these are actually six different classes, it remains to check that $\{\leftrightarrow, \widehat{\wedge}\}$, $\{\leftrightarrow, \widehat{\wedge}, -\}$ and $\{\leftrightarrow, \widehat{\wedge}, \neg\}$ generate different clones in \mathcal{BS}_0 , and the same with $\{\leftrightarrow, \sqcap\}$, $\{\leftrightarrow, \sqcap, -\}$ and $\{\leftrightarrow, \sqcap, \neg\}$.

Lemma 3.20. There exists an algebra $\mathbf{A} = (A, \wedge, \rightarrow, 1, 0) \in \mathcal{BS}_0$, and two sets $P, Q \subset A$ such that P is closed under $\leftrightarrow, \widehat{\wedge}, \sqcap$ but not under $-, \neg$ and Q is closed under $\leftrightarrow, \widehat{\wedge}, \sqcap, -$ but not under \neg .

Proof. Take $A = \{1, 2, 4, 3, 6, 12, 18, 0\} \subset \mathbb{Z}$ with the order $a \leq b \Leftrightarrow b|a$ (so 1 is maximal and 0 minimal). This ordering imposes the structure of a complete distributive lattice on A , so we can define $a \wedge b = \inf\{a, b\}$ and $a \rightarrow b = \bigvee\{c \in A : c \wedge a \leq b\}$. This makes $(A, \wedge, \rightarrow, 1, 0)$ a Brouwerian semilattice with zero.

Notice that with operations defined in this way, we get $r_{\mathcal{H}}(A) = \{1, 4, 18, 0\}$, $d_{\mathcal{H}}(A) = \{1, 2, 3, 6\}$. Define $Q = \{1, 2, 3, 6, 18\}$. This set does not contain $0 = \neg 1$, so it cannot be closed under \neg . On the other hand, it is easy to check that Q is closed under $\wedge, \rightarrow, r_{\mathcal{H}}, d_{\mathcal{H}}$, so it must also be closed under $\leftrightarrow, \widehat{\wedge}, \sqcap$. In Heyting algebras, $a - b \geq a$ holds, so Q is also closed under $-$.

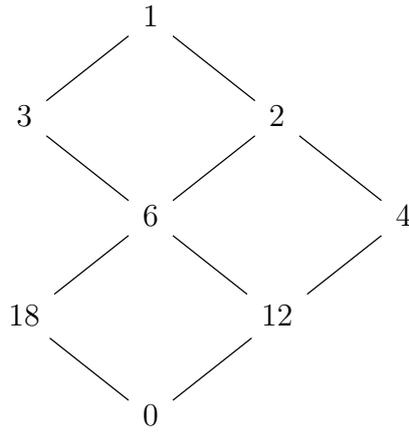


Figure 3.4: An algebra, which shows that the six sets of terms generate different clones.

Define $P = \{1, 6, 18\}$. Again, it is easy to check that P is closed under $\wedge, \rightarrow, r_{\mathcal{H}}, d_{\mathcal{H}}$, so it must be closed under $\leftrightarrow, \widehat{\wedge}, \sqcap$. However, $\neg 1 = 0 \notin P$ and $6 - 18 = 3 \notin P$. \square

Therefore, we have shown that there are six different classes of subreducts of \mathcal{BS}_0 , which satisfy our requirements. We are already familiar with two of them (EARS and EADS). To finish the proof of Theorem 3.1, it remains to check which of them are varieties. In the last chapter, we investigate the remaining four classes. Just like in EARS and EADS, we will first introduce some classes of algebras and later show that those are exactly the classes of subreducts that we are interested in.

Chapter 4

The remaining cases

4.1 Algebras with negation

Of the remaining four mixed type classes, we start with two containing negation. This is because equivalential algebras with negation have already been investigated ([23]) and because the last two cases will be reduced to them. First, we introduce some facts about equivalential algebras with zero:

Definition 4.1 ([23]). An algebra $(A, \cdot, 0)$ with one binary operation and one constant is called an *equivalential algebra with zero*, if (A, \cdot) is an equivalential algebra and the identity

$$x00yy \approx xyy$$

holds. Equivalential algebras with zero are term equivalent to *equivalential algebras with negation* (A, \cdot, \neg) by replacing the constant 0 by an unary operation $\neg x = x0$. Conversely, 0 is just $\neg 1$. The variety of equivalential algebras with zero is denoted by \mathcal{E}_0 .

Lemma 4.2 ([23]). Every equivalential algebra with zero \mathbf{A} has the same congruence lattice as its (\cdot) -reduct, hence \mathcal{E}_0 are congruence permutable Fregean. \mathcal{E}_0 has a largest sub-quasivariety that is not a variety, denoted \mathcal{E}_0^s and defined by adding the quasi-identity

$$0 \approx 1 \rightsquigarrow x \approx 1. \tag{q01}$$

\mathcal{E}_0^s is the class of $(\leftrightarrow, 0)$ -subreducts of Heyting algebras. The superscript “s” is short for separating, as those algebras separate 0 and 1.

The class of $(\leftrightarrow, 0)$ -subreducts is not a variety. There are two simple algebras in \mathcal{E}_0 : the reduct of a two-element Heyting algebra denoted $\mathbf{2}_0$, and a two-element equivalential algebra supplied with $0 = 1$ denoted $\mathbf{2}^0$. Algebra $\mathbf{2}^0$ cannot be a subreduct of a Heyting algebra, but $\mathbf{2}^0 \in H(\mathbf{2}_0 \times \mathbf{2}_0)$, so any variety that contains $\mathbf{2}_0$ must also contain $\mathbf{2}^0$. Notice that, in particular, the variety \mathcal{E}_0 is not primitive. The last point in Theorem 1.65 does not hold because $\{1\}$ is not a subuniverse of $\mathbf{2}_0$.

Now we can describe EARS with zero and EADS with zero. We will skip some of the details as the reasoning is just repeating the previous proofs with minor changes.

Definition 4.3. An algebra $(A, \cdot, \widehat{\wedge}, 0)$ with two binary operations and one constant is called an *equivalential algebra with regular semilattice and zero* if $(A, \cdot, \widehat{\wedge})$ is an EARS, $(A, \cdot, 0) \in \mathcal{E}_0$, and the following identities hold:

$$x00 = x\widehat{\wedge}x,$$

$$x\widehat{\wedge}0 = 0.$$

The first identity implies that the regularizations defined using $\widehat{\wedge}$ and 0 are equal. The second identity makes 0 the minimal element in the semilattice of regular elements.

Theorem 4.4. *The variety of EARS with zero coincides with the class of $(\leftrightarrow, \widehat{\wedge}, 0)$ -subreducts of Heyting algebras.*

Sketch of proof. It is easy to verify that Lemma 2.8 also holds for EARS with zero, and the proof of congruence orderability can also be repeated. Assume that we have a two-element EARS with zero with a universe $\{1, a\}$. Because equivalence on a two-element equivalential algebra is associative, we get $a = a00 = r(a)$ and $a\widehat{\wedge}0 = 0$. If $0 = 1$, then $1 = 0 = a\widehat{\wedge}0 = a\widehat{\wedge}1 = r(a) = a$ contradicts that we have a two-element universe. Hence, there is only one simple EARS with zero: the reduct of

a two-element Boolean algebra. We can also verify that all nontrivial algebras have unique extensions (as $\neg * = \neg r(*) = \neg 1$) and are locally finite (because condition (\dagger) holds). This means we can apply Theorem 2.15 and obtain the desired result. \square

Definition 4.5. An algebra $(A, \cdot, \sqcap, 0)$ with two binary operations and one constant is called an *equivalential algebra with dense semilattice and zero* if (A, \cdot, \sqcap) is an EADS, $(A, \cdot, 0) \in \mathcal{E}_0$, and the identity $x00x \approx x\sqcap x$, is satisfied.

Such algebra is called *separating* if it also satisfies (q_{01}) .

The identity $x00x \approx x\sqcap x$ just makes sure that $d(x)$ is the same operation, whether we define it using \sqcap or negation. As $r(x) = xd(x)$, also the regularizations are equal. EADS with zero is the unlucky case in which we cannot get a variety. Take $\mathbf{2}_0$ to be the $(\leftrightarrow, \sqcap, 0)$ -reduct of a two-element Heyting algebra. It has a universe $\{1, 0\}$ and \sqcap always returns 1. If we take $\mathbf{2}_0 \times \mathbf{2}_0$, then \sqcap is again trivial and the congruences are the same as on its equivalential reduct. The congruence $\Theta((1, 1), (0, 0))$ has two equivalence classes. If we take the quotient algebra we get an algebra isomorphic to $\mathbf{2}^0 = (\{1, a\}, \cdot, \widehat{\wedge}, 0)$ in which $0 = 1, x\sqcap y = 1$ and \cdot is the Boolean group operation. This algebra is certainly not a subreduct of any Heyting algebra, but it belongs to every variety containing $\mathbf{2}_0$.

Theorem 4.6. *The quasivariety of separating EADS with zero coincides with the class of $(\leftrightarrow, \sqcap, 0)$ -subreducts of Heyting algebras.*

Sketch of proof. The class \mathcal{V} of separating EADS with zero has only one simple algebra $\mathbf{2}_0$. In this algebra, both elements are regular and $\widehat{\wedge}$ is the minimum operation. Let \mathcal{W} be the quasivariety of all $(\leftrightarrow, \sqcap, 0)$ -subreducts of Heyting algebras. First, we verify that $\mathcal{W} \subset \mathcal{V}$ by checking that the required identities hold in Heyting algebras. Lemma 2.23 holds in \mathcal{V} , and this lets us repeat the proof of congruence orderability. Then, starting from the single simple algebra as a base case, we can perform the inductive proof that any finite member of \mathcal{V} is in \mathcal{W} .

If inclusion $\mathcal{W} \subsetneq \mathcal{V}$ is proper, then there must exist a quasi-identity q with n variables satisfied by all the algebras in \mathcal{W} but not by all in \mathcal{V} . Take $\mathbf{A} \in \mathcal{V}, \mathbf{A} \not\models q$. There are elements $x_1, \dots, x_n \in \mathbf{A}$ that falsify q . Take $\mathbf{B} \leq \mathbf{A}$ to be a subalgebra generated by these n elements. We have $\mathbf{B} \not\models q$ and moreover it is generated by

those n elements. However, \mathcal{V} is a subclass of all EADS with zero, which form a locally finite variety. This means \mathbf{B} is finite and by the aforementioned inductive proof $\mathbf{B} \in \mathcal{W}$. This gives us a contradiction.

This shows $\mathcal{W} = \mathcal{V}$ as needed. □

4.2 BEAN algebras

Now we will focus on the algebras arising as $(\leftrightarrow, -)$ -subreducts (we remind, that $x - y = x(\neg y)(\neg y)$). To easily distinguish them from the other cases, we pick a distinctive name. Looking at the behavior of clones in the previous chapter we can see that the operation “ $-$ ” gives rise to nearly the same terms as negation, so we call it almost-negation. That is also why we decided to denote this operation using a minus symbol. Our reducts contain *both equivalence and almost-negation*, hence we will call them BEANs.

We are not really interested in BEANs themselves. However, we need some knowledge about them before we consider $(\leftrightarrow, \widehat{\wedge}, -)$ -subreducts and $(\leftrightarrow, \sqcap, -)$ -subreducts.

In Heyting algebras, we can observe that fixing the second operand in almost-negation creates a polynomial that can be written as $x \mapsto x(\neg y)(\neg y)$. This polynomial is a modal operator (it is inflationary, idempotent, and preserves meets), just like a polynomial of the form $x \mapsto xzz$. Of course, in a Heyting algebra we can represent the first polynomial in the second form by taking $z = \neg y$, and the same applies to reducts. However, in a subreduct, we cannot always pick z such that $x - y = xzz$ as the required z could have been left outside the universe when taking a subalgebra. As terms $x \mapsto xzz$ play a crucial role in showing congruence orderability of equivalential algebras, we will define BEANs by adding identities that force $x - y$ to have similar properties.

Definition 4.7. Consider algebras of type $(\cdot, -)$ with two binary operations and define a unary term $r(x) = x - (x \cdot x)$. An algebra $\mathbf{A} = (A, \cdot, -)$ is called a *BEAN* if (A, \cdot, r) is an equivalential algebra with regularization and \mathbf{A} satisfies the following identities:

- N1. $x - x \approx r(x)$;
- N2. $(xy) - z \approx (x - z)(y - z)$;
- N3. $x - y \approx ((xr(x)) - r(y))r(x)$;
- N4. $(x - y)zz \approx (xzz) - y$;
- N5. $(x - (yz))r(y)r(y) \approx (x - z)r(y)r(y)$;
- N6. $(x - y) - z \approx (x - y)r(yz)r(yz)$;
- N7. $xy(x - z)(x - z) \approx xy$;
- N8. $(x - y) - z \approx (x - z) - y$;
- N9. $r(x - y) \approx r(x)$.

The class of all BEANs is denoted \mathcal{V}_{BEAN} .

We would like to point out that the above set of identities is not minimal. For example, N1 can be inferred from the remaining ones. We just picked a relatively large set of identities that are true in Heyting algebras and removed those that were not used in our proof, without any further refinement. As in previously analyzed classes, we would like to show that the class \mathcal{V}_{BEAN} coincides with the class of subreducts. The proof of congruence orderability in this case is harder than in EARS or EADS, we will need to repeat the reasoning behind the original proof for equivalential algebras [13].

After we show congruence orderability, there is one more problem: BEANs do not have unique extensions. After adding a new element $*$ to the universe, there may be more than one way of defining $* - y$ (we will show a small example in a moment, but it is possible to construct an arbitrarily large one). Because of that, we cannot use the inductive reasoning as in EARS or EADS to show that all finite BEANs are subreducts. We will instead show that every finite BEAN is a subreduct of an equivalential algebra with zero, which in turn is a subreduct of a Heyting algebra.

Lemma 4.8. $(\leftrightarrow, -)$ -reducts of Heyting algebras are BEANs.

Proof. In Heyting algebras, we can rewrite the identities to use only equivalence and a constant zero, as $x - y = x(y0)(y0)$. If we do that, then $x - 1 = x(10)(10) = x00 = r_{\mathcal{H}}(x)$, so the regularization operation defined using $-$ coincides with the Heyting one. After such a rewriting, the nine required identities quickly follow from the properties of \leftrightarrow and $r_{\mathcal{H}}$ (identities E1-E10 and R1-R7 are satisfied in \mathcal{H}). To make it even easier, it is useful to remember that $y0$ and 0 are regular elements. As an example we show the most complicated one, N3

$$\begin{aligned}
x - y &= x(y0)(y0) \\
&\stackrel{R4}{=} xr(x)r(x)(y0)(y0) \\
&\stackrel{R5}{=} xr(x)(y0)(y0)r(x) \\
&\stackrel{E6}{=} xr(x)(y000)(y000)r(x) \\
&= xr(x)(r(y)0)(r(y)0)r(x) \\
&= ((xr(x)) - r(y))r(x).
\end{aligned}$$

□

As mentioned in previous paragraphs, BEANs do not have unique extensions.

Example 4.9. Let $\mathbf{A} = (\{1, 0\}, \cdot, -)$ be the $(\leftrightarrow, -)$ -reduct of a two-element Heyting chain. This is an equivalential algebra with an additional binary operation that satisfies $x - y \approx x$. Take $\mathbf{B} = (\{1, *, 0, a, \neg a\}, \cdot, -)$ to be the $(\leftrightarrow, -)$ -reduct of the Heyting algebra $(\mathbf{2} \times \mathbf{2})^{\oplus}$ as shown in Figure 4.1.

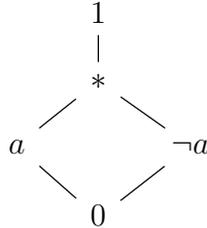


Figure 4.1: The Heyting algebra $(\mathbf{2} \times \mathbf{2})^{\oplus}$

In \mathbf{B} one has

$$x - y = \begin{cases} 1 & \text{if } x = *, y \neq 0 \\ x & \text{otherwise.} \end{cases}$$

The sets $\{1, *, a\}$, $\{1, *, 0\}$ are subuniverses of \mathbf{B} , they form subalgebras that are subdirectly irreducible. Both of those subalgebras have in turn a subalgebra isomorphic to \mathbf{A} which arises by removing $*$ from the universe. However, the two three-element algebras are not isomorphic. In the first algebra $* - y = 1$ holds for any y and in the second $* - 0 = *$. This shows that \mathbf{A} actually has two non-isomorphic extensions.

Intuitively, the operation \cdot does not provide a way to distinguish regular elements from each other (except 1), while $-$ also gives a special meaning to 0. The two possibilities arise because, in general, we do not know whether this special element 0 is in our universe. However, some BEANs have only one extension; we can be sure that 0 is missing if, for example, there is no regular element that is minimal in Fregean order.

The proof of congruence orderability will rely on the properties of filters in BEANs.

Definition 4.10. A *filter* on a BEAN $(A, \cdot, -)$ is a nonempty set $F \subset A$, such that for any $x, y \in A$ the following three “inference rules” are satisfied:

- $x \in F \Rightarrow xy \in F$,
- $x, xy \in F \Rightarrow y \in F$,
- $x \in F \Rightarrow x - y \in F$.

By convention we use $[X]$ to denote the smallest filter containing $X \subset A$, and for a single element we write $[a] = [\{a\}]$. If only the first two inference rules hold, then the set is a filter on the equivalential reduct (as in Definition 1.58). It is easy to notice that $[X]$ contains X , and all elements which can be obtained by using the inference rules finitely many times.

Just like in equivalential algebras, there is a one-to-one correspondence between filters and congruences.

Lemma 4.11. Let $\mathbf{A} = (A, \cdot, -)$ be a BEAN. The relation $\theta \in A \times A$ is a congruence if and only if $\theta = \{(x, y) : xy \in F\}$ for some filter F .

Proof. If $\theta \in \text{Con}(\mathbf{A})$, then it is also a congruence on the equivalential reduct, so $1/\theta$ is an equivalential filter. But for any $x \equiv_{\theta} 1$ we have $x - y \equiv_{\theta} 1 - y = (xx) - y \stackrel{N2}{=} (x - y)(x - y) = 1$, so $1/\theta$ is a BEAN filter.

Assume now that F is a filter and $\theta = \{(x, y) : xy \in F\}$. As a BEAN filter is also an equivalential filter, θ is a congruence on the equivalential reduct. We only need to check if it preserves $-$. For any $xy \in F, z \in A$, we have $(x - z)(y - z) = (xy) - z \in F$. To show that also $(z - x)(z - y) \in F$, first observe that $r(xy) = (xy) - 1 \in F$. Now we can write

$$\begin{aligned}
F \ni 1 &= (z - x)(z - x)r(xy)r(xy) \\
&\stackrel{E10}{=} ((z - x)r(xy)r(xy))((z - x)r(xy)r(xy)) \\
&\stackrel{N6}{=} ((z - x) - y)((z - x)r(xy)r(xy)) \\
&\stackrel{N8}{=} ((z - y) - x)((z - x)r(xy)r(xy)) \\
&\stackrel{N6}{=} ((z - y)r(xy)r(xy))((z - x)r(xy)r(xy)) \\
&\stackrel{E10}{=} (z - y)(z - x)r(xy)r(xy),
\end{aligned}$$

and using the second inference rule $(z - y)(z - x)r(xy) \in F$ and $(z - y)(z - x) \in F$, what finishes the proof. \square

Lemma 4.12. Let $\mathbf{A} = (A, \cdot, -)$ be a BEAN. For any $a, b, c \in A, b \in [a]$, we have $cbbaa = caa$.

Proof. Fix $a \in A$ and consider the set $B_a = \{b \in A : \forall c \in A \text{ } cbbaa = caa\}$. We will show that it is a filter. We take any $b, bf \in B, e \in A$ and we will verify that elements $bee, b - e, f$ are also in B_a :

$$\begin{aligned}
c(bee)(bee)aa &\stackrel{E8}{=} caa(bee)(bee) \\
&= cbbaa(bee)(bee) \\
&\stackrel{E8}{=} cbb(bee)(bee)aa \\
&\stackrel{E3}{=} cbbaa \\
&= caa, \quad \text{so } bee \in B_a;
\end{aligned}$$

$$\begin{aligned}
c(b-e)(b-e)aa &\stackrel{E8}{=} caa(b-e)(b-e) \\
&= cbbaa(b-e)(b-e) \\
&\stackrel{E8}{=} cbb(b-e)(b-e)aa \\
&\stackrel{N7}{=} cbbaa \\
&= caa, \quad \text{so } b-e \in B_a;
\end{aligned}$$

$$\begin{aligned}
cffaa &\stackrel{E8}{=} caaff \\
&= cbbaaff \\
&\stackrel{E8}{=} cbbffaa \\
&\stackrel{E9}{=} cbb(bf)(bf)aa \\
&\stackrel{E8}{=} cbbaa(bf)(bf) \\
&= caa(bf)(bf) \\
&\stackrel{E8}{=} c(bf)(bf)aa \\
&= caa, \quad \text{so } f \in B_a.
\end{aligned}$$

As B_a is a filter and $a \in B_a$ we have $[a] \subset B_a$, which completes the proof. \square

Definition 4.13. For any BEAN $\mathbf{A} = (A, \cdot, -)$, we define an algebra $\mathcal{S}(\mathbf{A})$ as the set of functions $A \mapsto A$ with an operation of function composition, which is generated by the set $\{x \mapsto xyy : y \in A\} \cup \{x \mapsto x - y : y \in A\}$.

Lemma 4.14. Let $\mathbf{A} = (A, \cdot, -)$ be a BEAN and $a \in A$. Then:

- (a) $\mathcal{S}(\mathbf{A}) = (S, \circ)$ is a semilattice of homomorphisms $A \mapsto A$;
- (b) $\{f(a) : f \in \mathcal{S}(\mathbf{A})\} \subset [a]$;
- (c) The operation \cdot is associative in the set $\{f(a) : f \in \mathcal{S}(\mathbf{A})\}$.

Proof. (a): Let f, g be functions from the set of generators of $\mathcal{S}(\mathbf{A})$. Those functions are idempotent (either by E6 or N6), homomorphisms (either by E10 or N2), and their composition is commutative (either by E8, N4, or N8). As S is generated by such functions, these three properties extend to the entire set S . The composition of unary functions is always associative; therefore, $\mathcal{S}(\mathbf{A})$ is a semilattice.

(b): The filter $[a]$ is closed under the generators of $\mathcal{S}(\mathbf{A})$, so it must also be closed under their compositions.

(c): Let $x, y, z \in \{f(a) : f \in \mathcal{S}(\mathbf{A})\}$. We have $f(a) = x$ for some $f \in \mathcal{S}(\mathbf{A})$, and by the commutativity of $\mathcal{S}(\mathbf{A})$, the equality $xaa = f(a)aa = f(aaa) = f(a) = x$ holds. Similarly, $y = yaa, z = zaa$. By the previous point, $y \in [a]$, and we can apply Lemma 4.12 for element y . We now calculate

$$\begin{aligned}
xyz &= (xaa)(yaa)(zaa) \\
&\stackrel{E10}{=} xyzaa \\
&= xyzyyaa \\
&\stackrel{E7}{=} x(yz)yyaa \\
&= x(yz)aa \\
&\stackrel{E10}{=} (xaa)((yaa)(zaa)) \\
&= x(yz).
\end{aligned}$$

This shows associativity. □

Lemma 4.15. Let $\mathbf{A} = (A, \cdot, -)$ be a BEAN. For any $a \in A, b \in [a]$ there exist functions $g_1, \dots, g_m \in \mathcal{S}(\mathbf{A})$ such that

$$baa = g_1(a) \cdot g_2(a) \cdot \dots \cdot g_m(a).$$

Moreover, $[a] = [b] \Rightarrow a = b$.

Proof. First, we want to show that the set

$$F = \{b : \exists_{g_1, \dots, g_m \in \mathcal{S}(\mathbf{A})} baa = g_1(a) \cdot g_2(a) \cdot \dots \cdot g_m(a)\}$$

is a filter. It is easy to see that $F \subset [a]$, as for any $b \in F$

$$b \equiv_{\Theta(1,a)} baa = g_1(a) \cdot g_2(a) \cdot \dots \cdot g_i(a) \equiv_{\Theta(1,a)} 1.$$

Assume that $b \in F, c \in A$, therefore $baa = g_1(a) \cdot \dots \cdot g_m(a)$. Define the functions $h'(x) = xcc, h''(x) = x - c$, which are elements of $\mathcal{S}(\mathbf{A})$. As $\mathcal{S}(\mathbf{A})$ is closed under

function composition, $g'_i = h' \circ g_i$ and $g''_i = h'' \circ g_i$ are also its elements for any $i \in \{1, \dots, m\}$. We now write

$$\begin{aligned} bccaa &\stackrel{E8}{=} baacc \\ &= (g_1(a) \cdot \dots \cdot g_m(a))cc \\ &= h'(g_1(a) \cdot \dots \cdot g_m(a)) \\ &= g'_1(a) \cdot \dots \cdot g'_m(a), \end{aligned}$$

$$\begin{aligned} (b-c)aa &\stackrel{N4}{=} (baa) - c \\ &= (g_1(a) \cdot \dots \cdot g_m(a)) - c \\ &= h''(g_1(a) \cdot \dots \cdot g_m(a)) \\ &= g''_1(a) \cdot \dots \cdot g''_m(a). \end{aligned}$$

The last equality holds because, by the previous lemma, h', h'' are homomorphisms. This shows that $bcc, b-c \in F$. Similarly, if $b \in F, bc \in F$, then there exist functions $g_1, \dots, g_m, f_1, \dots, f_n$ such that $baa = g_1(a) \cdot \dots \cdot g_m(a)$ and $bcaa = f_1(a) \cdot \dots \cdot f_n(a)$. We have

$$f_1(a) \cdot \dots \cdot f_n(a) = bcaa = (baa)(caa),$$

$$\begin{aligned} (f_1(a) \cdot \dots \cdot f_n(a)) \cdot (g_1(a) \cdot \dots \cdot g_m(a)) &= (f_1(a) \dots f_n(a))(baa) \\ &= caa(baa)(baa) \\ &= c(baa)(baa)aa. \end{aligned}$$

We assumed, that $b \in F$, so also $baa \in F$ and we can apply Lemma 4.12 on the right-hand side of the above equality. On the left-hand side we can ignore the parentheses by Lemma 4.14(c), hence

$$f_1(a) \cdot \dots \cdot f_n(a) \cdot g_1(a) \cdot \dots \cdot g_m(a) = caa.$$

This shows that F is a filter. We also have $aaa = a = a11$, so $a \in F, [a] = F$.

If $[a] = [b]$, then we apply the first part twice, so

$$abb = f_1(b) \cdot f_2(b) \cdot \dots \cdot f_n(b)$$

$$baa = g_1(a) \cdot g_2(a) \cdot \dots \cdot g_m(a)$$

for some $f_1, \dots, f_n, g_1, \dots, g_m \in \mathcal{S}(\mathbf{A})$.

Combining those two equalities we can write

$$\begin{aligned}
abb &\stackrel{E1}{=} aaabb \stackrel{E8}{=} abbaa \\
&= f_1(b) \cdot f_2(b) \cdot \dots \cdot f_n(b) \cdot a \cdot a \\
&= f_1(baa) \cdot f_2(baa) \cdot \dots \cdot f_n(baa) \\
&\stackrel{E1}{=} f_1(bbaa) \cdot f_2(bbaa) \cdot \dots \cdot f_n(bbaa) \\
&\stackrel{E8}{=} f_1(baabb) \cdot f_2(baabb) \cdot \dots \cdot f_n(baabb) \\
&= f_1(g_1(a) \cdot \dots \cdot g_m(a)bb) \cdot f_2(g_1(a) \cdot \dots \cdot g_m(a)bb) \cdot \dots \cdot f_n(g_1(a) \cdot \dots \cdot g_m(a)bb) \\
&= f_1(g_1(abb) \cdot \dots \cdot g_m(abb)) \\
&\quad \cdot f_2(g_1(abb) \cdot \dots \cdot g_m(abb)) \\
&\quad \cdot \dots \cdot f_n(g_1(abb) \cdot \dots \cdot g_m(abb)) \\
&= f_1\left(g_1(f_1(b) \cdot \dots \cdot f_n(b)) \cdot \dots \cdot g_m(f_1(b) \cdot \dots \cdot f_n(b))\right) \\
&\quad \cdot f_2\left(g_1(f_1(b) \cdot \dots \cdot f_n(b)) \cdot \dots \cdot g_m(f_1(b) \cdot \dots \cdot f_n(b))\right) \\
&\quad \cdot \dots \cdot f_n\left(g_1(f_1(b) \cdot \dots \cdot f_n(b)) \cdot \dots \cdot g_m(f_1(b) \cdot \dots \cdot f_n(b))\right) \\
&= (f_1 \circ g_1 \circ f_1)(b) \cdot \dots \cdot (f_n \circ g_m \circ f_n)(b).
\end{aligned}$$

Elements in the last line are of the form $(f_i \circ g_j \circ f_k)(b)$ for all $i, k \in \{1, \dots, n\}, j \in \{1, \dots, m\}$. By Lemma 4.14 (c), the operation \cdot is associative among these elements. Moreover, from Lemma 4.14 (a), the composition is commutative on $\mathcal{S}(\mathbf{A})$, so we have $(f_i \circ g_j \circ f_k)(b) = (f_k \circ g_j \circ f_i)(b)$. This means that $(f_i \circ g_j \circ f_k)(b) \cdot (f_k \circ g_j \circ f_i)(b) = 1$, so we can reduce all elements with $i \neq k$. By idempotence and commutativity of composition in $\mathcal{S}(\mathbf{A})$ we simplify it further as $(f_i \circ g_j \circ f_i)(b) = (f_i \circ g_j)(b)$ and we obtain

$$abb = (f_1 \circ g_1)(b) \cdot (f_1 \circ g_2)(b) \cdot \dots \cdot (f_1 \circ g_m)(b) \cdot (f_2 \circ g_1)(b) \cdot \dots \cdot (f_n \circ g_m)(b).$$

On the other hand,

$$\begin{aligned}
baa &= baabb \\
&= g_1(a) \cdot g_2(a) \cdot \dots \cdot g_m(a) \cdot b \cdot b \\
&= g_1(abb) \cdot g_2(abb) \cdot \dots \cdot g_m(abb) \\
&= g_1(f_1(b) \cdot f_2(b) \cdot \dots \cdot f_n(b)) \\
&\quad \cdot g_2(f_1(b) \cdot f_2(b) \cdot \dots \cdot f_n(b)) \\
&\quad \cdot \dots \cdot g_n(f_1(b) \cdot f_2(b) \cdot \dots \cdot f_n(b)) \\
&= (g_1 \circ f_1)(b) \cdot \dots \cdot (g_m \circ f_n)(b) \\
&= abb.
\end{aligned}$$

This proves $abb = baa$, but by properties of equivalential algebras

$$\begin{aligned}
abb(baa) &\stackrel{E5}{=} (b(ab))(a(ab)) \\
&\stackrel{E2}{=} ba(ab)(ab) \\
&\stackrel{E1}{=} ab,
\end{aligned}$$

so $a \in [b], b \in [a]$ implies $a = b$. □

This completes the proof of congruence orderability. Because \mathcal{E} are reducts of BEAN algebras, \mathcal{V}_{BEAN} is congruence permutable and 1-regular. This lets us use Lemma 1.62 and Theorem 1.65 (N4 and E10 imply condition (\dagger)) to conclude the following:

Corollary 4.16. *The variety \mathcal{V}_{BEAN} is Fregean, congruence permutable, locally finite, and primitive. Every subdirectly irreducible BEAN \mathbf{A} has the element $*$ such that $\mu_{\mathbf{A}} = \{(x, y) : x = y \text{ or } \{x, y\} = \{1, *\}\}$ and $\text{Univ}(\mathbf{A}) \setminus \{*\}$ is a subuniverse.*

In an equivalential algebra with regularization (and in BEANs) we have the identity R4: $x \approx (xr(x))r(x)$. This implies that the function $f : x \mapsto (r(x), xr(x))$ is injective. We can treat each element in an algebra as a pair $(r(x), d(x))$ ($d(x) = xr(x)$ is the unary operation that we call densification, because in Heyting algebras it is equal to $d_{\mathcal{H}}$). For any $(A, \cdot, r) \in \mathcal{E}_r$ the three sets $d(A) = \{d(x) : x \in A\}$, $\{x \in A : r(x) = 1\}$, $\{x \in A : x = d(x)\}$ are equal as, shown in Lemma 1.93. If we take

$a \in r(A), b \in d(A)$, then $ab \in A$ and $r(ab) = r(a)r(b) = a$, but $d(ab) = abr(ab) = baa$, which is not necessarily equal to b . This lets us conclude that the image $f(A)$ is composed of pairs $(a, b) \in r(A) \times d(A)$ such that $baa = b$.

Now, our approach is to remove the need for the operation $-$ by introducing a zero element to the algebra. By N3, in any BEAN the value of $x - y$ does not depend on the dense part of y . We will extend a BEAN in such a manner that for any regular element y there is also another regular element \bar{y} satisfying $x - y = x \cdot \bar{y} \cdot \bar{y}$. Then because of above observation, we also need to add elements of the form $z\bar{y}$ where $z \in d(A)$ and $z\bar{y}\bar{y} = z$.

Definition 4.17. Let $\mathbf{A} = (A, \cdot, -)$ be a BEAN. Define $R(A^0) = r(A) \cup \{\bar{a} : a \in r(A)\}$, we extend operation \cdot to this set by setting for any $a, c \in r(A)$

$$a \cdot \bar{c} = \bar{a} \cdot c = \overline{ac},$$

$$\bar{a} \cdot \bar{c} = ac.$$

To better grasp what we mean remember that $(r(A), \cdot)$ is a Boolean group, so it is also a vector space over the two-element field. What we do here is expand this vector space by one dimension. If B is a basis of $r(A)$, then $B \cup \{\bar{1}\}$ is a basis of $R(A^0)$. For any $a \in r(A)$ we take

$$\pi_a : d(A) \ni b \mapsto baa \in d(A),$$

$$\pi_{\bar{a}} : d(A) \ni b \mapsto b - a \in d(A).$$

We define a new algebra $\mathbf{A}^0 = (A^0, \odot, 0)$ with one binary operation and one constant in the following way

$$A^0 = \{(a, b) : a \in R(A^0), b \in d(A), \pi_a(b) = b\};$$

$$0 = (\bar{1}, 1);$$

and for any $a, c \in R(A^0), b, e \in d(A)$

$$(a, b) \odot (c, e) = (ac, \pi_c(b)\pi_a(e)).$$

Lemma 4.18. For any BEAN \mathbf{A} , the algebra \mathbf{A}^0 is an equivalential algebra with zero.

Proof. First, we need to see some properties of the π functions. To do that, we combine some of the identities from the definition of BEANs with their counterparts for functions of the form $x \mapsto xr(y)r(y)$ in \mathcal{E}_r . We will show that the following properties hold for any $a, c \in R(A^0), b, e \in d(A)$:

P1. $(a, b) \in A^0 \Rightarrow \pi_a(b) = b$;

P2. $(\pi_a \circ \pi_a)(b) = \pi_a(b)$ (idempotence);

P3. $(\pi_a \circ \pi_c)(b) = (\pi_c \circ \pi_a)(b)$ (commutativity);

P4. $\pi_a(be) = \pi_a(b)\pi_a(e)$ and in particular $\pi_a(1) = 1$ (being a homomorphism);

P5. $(\pi_{ac} \circ \pi_a)(b) = (\pi_c \circ \pi_a)(b)$ (generalized E9);

P6. $(a, b) \in A^0 \Rightarrow \pi_{ac}(b) = (\pi_c \circ \pi_a)(b) = \pi_c(b)$;

P7. if \mathbf{A} is subdirectly irreducible and $\pi_a(b) = *$ then $b = *, \pi_a = \text{Id}_A$.

The first property follows from the definition of A^0 . The second, third and fourth follow from Lemma 4.14 (a) as $\pi_a, \pi_c \in \mathcal{S}(A)$. The fifth is done by checking all possible cases. If $a, c \in r(A)$, then

$$\begin{aligned} (\pi_{ac} \circ \pi_a)(b) &= br(a)r(a)r(ac)r(ac) \\ &\stackrel{R2}{=} br(a)r(a)(r(a)r(c))(r(a)r(c)) \\ &\stackrel{E9}{=} br(a)r(a)r(c)r(c) \\ &= (\pi_c \circ \pi_a)(b). \end{aligned}$$

If $a \in r(A), c = \bar{y}, y \in r(A)$, we have

$$\begin{aligned} (\pi_{ac} \circ \pi_a)(b) &\stackrel{P3}{=} (\pi_a \circ \pi_{a\bar{y}})(b) \\ &= (b - (ay))r(a)r(a) \\ &\stackrel{N5}{=} (b - y)r(a)r(a) \\ &= (\pi_a \circ \pi_{\bar{y}})(b) \\ &\stackrel{P3}{=} (\pi_c \circ \pi_a)(b). \end{aligned}$$

If $a = \bar{x}, x \in r(A), c \in r(A)$, then it follows that

$$\begin{aligned}
(\pi_{ac} \circ \pi_a)(b) &= (b - x) - (cx) \\
&\stackrel{N6}{=} (b - x)r(cxx)r(cxx) \\
&\stackrel{R3}{=} (b - x)r(c)r(c) \\
&= (\pi_c \circ \pi_a)(b).
\end{aligned}$$

And in the last case $a = \bar{x}, c = \bar{y}, x \in r(A), y \in r(A)$, so

$$\begin{aligned}
(\pi_{ac} \circ \pi_a)(b) &\stackrel{P3}{=} (\pi_a \circ \pi_{xy})(b) \\
&= (b - x)r(xy)r(xy) \\
&\stackrel{N6}{=} (b - x) - y \\
&= (\pi_{\bar{y}} \circ \pi_{\bar{x}})(b) \\
&= (\pi_c \circ \pi_a)(b).
\end{aligned}$$

For the sixth property, it is enough to notice that $\pi_{ac}(b) \stackrel{P1}{=} (\pi_{ac} \circ \pi_a)(b) \stackrel{P5}{=} (\pi_c \circ \pi_a)(b) \stackrel{P1}{=} \pi_c(b)$. In a subdirectly irreducible algebra, if the kernel of π_a is nontrivial, then $\pi_a(*) = \pi_a(1) = 1$, and because this is an idempotent homomorphism, $* \notin \pi_a(A)$. Hence, the only possibility for equality $\pi_a(b) = *$ to hold is π_a having a trivial kernel, as π_a is idempotent, it must be the identity mapping. If this is the case, then $b = \pi_a(b) = *$ is showing P7.

We will now apply P1-P7 to show that the identities of \mathcal{E}_0 hold in \mathbf{A}^0 . Of course, all the complexity is in the second coordinate, as \odot operates on the first coordinate just like the \cdot . Choose any $(a, x), (b, y), (c, z) \in A^0$, then

$$\begin{aligned}
(a, x) \odot (a, x) \odot (b, y) &= (aa, \pi_a(x)\pi_a(x)) \odot (b, y) \\
&\stackrel{E1}{=} (1, 1) \odot (b, y) \\
&= (1b, \pi_1(y)\pi_b(1)) \\
&= (b, (y11)\pi_b(1)) \\
&\stackrel{E1}{=} (b, y\pi_b(1)) \\
&\stackrel{P4}{=} (b, y),
\end{aligned}$$

which shows that E1 holds. To show the remaining equalities, we first perform some simplifications.

$$\begin{aligned}
& (a, x) \odot (c, z) \odot (c, z) \\
&= (ac, \pi_c(x)\pi_a(z)) \odot (c, z) \\
&= (acc, \pi_c(\pi_c(x)\pi_a(z))\pi_{ac}(z)) \\
&\stackrel{P4}{=} (acc, (\pi_c \circ \pi_c)(x) \cdot (\pi_c \circ \pi_a)(z) \cdot \pi_{ac}(z)) \\
&\stackrel{P2}{=} (acc, \pi_c(x) \cdot (\pi_c \circ \pi_a)(z) \cdot \pi_{ac}(z)) \\
&\stackrel{P6}{=} (acc, \pi_c(x) \cdot \pi_a(z) \cdot \pi_a(z)) \\
&= (a, \pi_c(x)\pi_a(z)\pi_a(z)),
\end{aligned}$$

and we define $u = \pi_c(x)\pi_a(z)\pi_a(z) \in d(A)$. Notice that $(a, u) \in A^0$ as

$$\begin{aligned}
\pi_a(u) &\stackrel{P4}{=} (\pi_a \circ \pi_c)(x) \cdot (\pi_a \circ \pi_a)(z) \cdot (\pi_a \circ \pi_a)(z) \\
&\stackrel{P2}{=} (\pi_a \circ \pi_c)(x) \cdot \pi_a(z) \cdot \pi_a(z) \\
&\stackrel{P6}{=} u.
\end{aligned}$$

Now, we compute the left-hand side of E3:

$$\begin{aligned}
& (a, x) \odot (b, y) \odot ((a, x) \odot (c, z) \odot (c, z)) \odot ((a, x) \odot (c, z) \odot (c, z)) \\
&= (ab, \pi_b(x)\pi_a(y)) \odot (a, u) \odot (a, u) \\
&= (aba, \pi_a(\pi_b(x)\pi_a(y)) \cdot \pi_{ab}(u)) \odot (a, u) \\
&\stackrel{P4}{=} (b, (\pi_a \circ \pi_b)(x) \cdot (\pi_a \circ \pi_a)(y) \cdot \pi_{ab}(u)) \odot (a, u) \\
&\stackrel{P2}{=} (b, (\pi_a \circ \pi_b)(x) \cdot \pi_a(y) \cdot \pi_{ab}(u)) \odot (a, u) \\
&\stackrel{P6}{=} (b, \pi_b(x) \cdot \pi_a(y) \cdot \pi_b(u)) \odot (a, u) \\
&= (ba, \pi_a(\pi_b(x) \cdot \pi_a(y) \cdot \pi_b(u)) \cdot \pi_b(u)) \\
&\stackrel{P4}{=} (ab, (\pi_a \circ \pi_b)(x) \cdot (\pi_a \circ \pi_a)(y) \cdot (\pi_a \circ \pi_b)(u) \cdot \pi_b(u)) \\
&\stackrel{P6}{=} (ab, \pi_b(x) \cdot (\pi_a \circ \pi_a)(y) \cdot \pi_b(u) \cdot \pi_b(u)) \\
&\stackrel{P2}{=} (ab, \pi_b(x) \cdot \pi_a(y) \cdot \pi_b(u) \cdot \pi_b(u)).
\end{aligned}$$

The right-hand side of E3 is $(a, x) \odot (b, y) = (ab, \pi_b(x)\pi_a(y))$. Similarly, we

approach E2:

$$\begin{aligned}
& (a, x) \odot (b, y) \odot (c, z) \odot (c, z) \\
&= (ab, \pi_b(x)\pi_a(y)) \odot (c, z) \odot (c, z) \\
&= (abc, \pi_c(\pi_b(x)\pi_a(y))\pi_{ab}(z)) \odot (c, z) \\
&= (abcc, \pi_c(\pi_c(\pi_b(x)\pi_a(y))\pi_{ab}(z))\pi_{abc}(z)) \\
&\stackrel{P4}{=} (abcc, (\pi_c \circ \pi_c \circ \pi_b)(x) \cdot (\pi_c \circ \pi_c \circ \pi_a)(y) \cdot (\pi_c \circ \pi_{ab})(z) \cdot \pi_{abc}(z)) \\
&\stackrel{P2}{=} (abcc, (\pi_c \circ \pi_b)(x) \cdot (\pi_c \circ \pi_a)(y) \cdot (\pi_c \circ \pi_{ab})(z) \cdot \pi_{abc}(z)) \\
&\stackrel{P6}{=} (abcc, (\pi_c \circ \pi_b)(x) \cdot (\pi_c \circ \pi_a)(y) \cdot \pi_{ab}(z) \cdot \pi_{ab}(z)),
\end{aligned}$$

$$\begin{aligned}
& ((a, x) \odot (c, z)) \odot ((b, y) \odot (c, z)) \\
&= (ac, \pi_c(x)\pi_a(z)) \odot (bc, \pi_c(y)\pi_b(z)) \\
&= ((ac)(bc), \pi_{bc}(\pi_c(x)\pi_a(z))\pi_{ac}(\pi_c(y)\pi_b(z))) \\
&\stackrel{P4}{=} ((ac)(bc), ((\pi_{bc} \circ \pi_c)(x) \cdot (\pi_{bc} \circ \pi_a)(z)) \cdot ((\pi_{ac} \circ \pi_c)(y) \cdot (\pi_{ac} \circ \pi_b)(z))) \\
&\stackrel{P5}{=} ((ac)(bc), ((\pi_b \circ \pi_c)(x) \cdot (\pi_{bc} \circ \pi_a)(z)) \cdot ((\pi_a \circ \pi_c)(y) \cdot (\pi_{ac} \circ \pi_b)(z))) \\
&\stackrel{P6}{=} ((ac)(bc), ((\pi_b \circ \pi_c)(x) \cdot (\pi_b \circ \pi_a)(z)) \cdot ((\pi_a \circ \pi_c)(y) \cdot (\pi_a \circ \pi_b)(z))) \\
&\stackrel{E2}{=} (abcc, (\pi_c \circ \pi_b)(x) \cdot (\pi_c \circ \pi_a)(y) \cdot (\pi_a \circ \pi_b)(z) \cdot (\pi_a \circ \pi_b)(z)).
\end{aligned}$$

Therefore for E2 to hold in \mathbf{A}^0 , we need

$$\begin{aligned}
& (\pi_c \circ \pi_b)(x) \cdot (\pi_c \circ \pi_a)(y) \cdot \pi_{ab}(z) \cdot \pi_{ab}(z) \\
&= (\pi_c \circ \pi_b)(x) \cdot (\pi_c \circ \pi_a)(y) \cdot (\pi_a \circ \pi_b)(z) \cdot (\pi_a \circ \pi_b)(z)
\end{aligned} \tag{4.1}$$

to hold in \mathbf{A} . Similarly, if

$$\pi_b(x) \cdot \pi_a(y) \cdot \pi_b(u) \cdot \pi_b(u) = \pi_b(x) \cdot \pi_a(y) \tag{4.2}$$

holds in \mathbf{A} , then E3 is true in \mathbf{A}^0 . Each of the two equations is actually 8 different identities (depending on whether a, b, c are “real” regular elements or the negated versions that we added to the larger universe) with 6 variables each. We will postpone proof of them for a moment to minimize abuse of notation.

To prove \mathbf{A}^0 is an equivalential algebra with zero we need to show that $0 = (\bar{1}, 1)$ has the property required by Definition 4.1 (for any $p, q \in \mathbf{A}^0$ we must have $p00qq =$

$p00$). Using $x = d(x)$, we get

$$\begin{aligned}
& (a, x) \odot 0 \odot 0 \\
&= (a, x) \odot (\bar{1}, 1) \odot (\bar{1}, 1) \\
&= (a\bar{1}, (x-1)\pi_a(1)) \odot (\bar{1}, 1) \\
&= (\bar{a}, r(x)) \odot (\bar{1}, 1) \\
&= (\bar{a}, r(d(x))) \odot (\bar{1}, 1) \\
&= (\bar{a}, 1) \odot (\bar{1}, 1) \\
&= (\bar{a}\bar{1}, \pi_{\bar{1}}(1)\pi_{\bar{a}}(1)) \\
&= (a, 1),
\end{aligned}$$

so

$$\begin{aligned}
& (a, x) \odot 0 \odot 0 \odot (b, y) \odot (b, y) \\
&= (a, 1) \odot (b, y) \odot (b, y) \\
&= (ab, \pi_b(1)\pi_a(y)) \odot (b, y) \\
&= (ab, \pi_a(y)) \odot (b, y) \\
&= (abb, \pi_a(y)\pi_{ab}(y)) \\
&= (a, \pi_a(y)\pi_{ab}(y)) \\
&\stackrel{P1}{=} (a, \pi_a(y) \cdot (\pi_{ab} \circ \pi_b)(y)) \\
&\stackrel{P6}{=} (a, \pi_a(y) \cdot (\pi_a \circ \pi_b)(y)) \\
&\stackrel{P1}{=} (a, \pi_a(y) \cdot \pi_a(y)) \\
&= (a, 1) \\
&= (a, x) \odot 0 \odot 0.
\end{aligned}$$

At this point, it only remains to show the equalities (4.1) and (4.2). We will now use the “proof by the lack of a counterexample” method. Assume to the contrary that one of equalities is not true in BEANs. Using the reasoning described in Lemma 1.92 there exists a subdirectly irreducible counterexample algebra \mathbf{A} and elements $a, b, c \in r(\mathbf{A}), x, y, z \in d(\mathbf{A})$ such that one side of the equality is equal to 1 and the other is $*$.

We recall from Corollary 1.67, that in a subdirectly irreducible equivalential algebra the equality $vw = *$ holds if and only if $\{v, w\} = \{1, *\}$, and the equality $vw w = *$ holds only if $v = *, w \in \{1, *\}$. This property is also true in BEANs.

If any side of the equality (4.1) is equal to $*$, then we get $(\pi_c \circ \pi_b)(x)(\pi_c \circ \pi_a)(y) = *$. Hence, one of the elements $(\pi_c \circ \pi_b)(x), (\pi_c \circ \pi_a)(y)$ must be equal to $*$. By P7, this means that either $\pi_c \circ \pi_b$ or $\pi_c \circ \pi_a$ is an identity mapping. This in turn means that one of π_a, π_b is identity. In either case, $(\pi_{ab} \circ \text{Id})(z) = (\pi_a \circ \pi_b)(z)$ by P5, so the equality (4.1) holds.

Now we look at the equality (4.2). No matter which side we assume is equal to $*$, we obtain $\{\pi_b(x), \pi_a(y)\} = \{1, *\}$. If $\pi_b(x) = 1$, then

$$\begin{aligned} \pi_b(u) &\stackrel{P4}{=} (\pi_b \circ \pi_c)(x) \cdot (\pi_b \circ \pi_a)(z) \cdot (\pi_b \circ \pi_a)(z) \\ &= \pi_c(1) \cdot (\pi_b \circ \pi_a)(z) \cdot (\pi_b \circ \pi_a)(z) \\ &= 1 \cdot (\pi_b \circ \pi_a)(z) \cdot (\pi_b \circ \pi_a)(z) \\ &= 1 \end{aligned}$$

and the equality does hold. Now, assume $\pi_b(x) = *$ and remember that the image of $*$ under any homomorphism is equal to either $*$ or 1. If we consider

$$\pi_b(u) = \pi_c(*) \cdot (\pi_b \circ \pi_a)(z) \cdot (\pi_b \circ \pi_a)(z),$$

then $\pi_c(*) \in \{1, *\}$ and so does $\pi_b(u)$. Hence, the left-hand side is reduced either to $*1** = *$ or $*111 = *$. In any case it is equal to the right-hand side. \square

Theorem 4.19. *The class of BEANs is the class of $(\leftrightarrow, -)$ -subreducts of Heyting algebras.*

Proof. Every $(\leftrightarrow, -)$ -subreduct is a BEAN, as the definition of BEAN uses only identities that are true in Heyting algebras. Consider any BEAN \mathbf{A} and construct \mathbf{A}^0 . By definition, the zero element in \mathbf{A}^0 is the pair $(\bar{1}, 1)$, which is not equal to the unit element $(1, 1)$. Hence (q_{01}) holds in \mathbf{A}^0 and, by Lemma 4.2, it is a $(\leftrightarrow, 0)$ -subreduct of a Heyting algebra \mathbf{H} . We extend the language of \mathbf{A}^0 by adding $(a, b) - (c, e) = (a, b) \odot (0 \odot (c, e)) \odot (0 \odot (c, e))$. The operation $-$ is defined by equivalence and zero, and we used the same definition as in Heyting algebras, so

after such an extension \mathbf{A}^0 is still a subreduct of \mathbf{H} . It remains to show that \mathbf{A} is a subreduct of \mathbf{A}^0 , and it will also be a subreduct of \mathbf{H} .

First, we will simplify the definition of $-$. If $(c, e) \in A^0$, then

$$\begin{aligned}
0 \odot (c, e) &= (\bar{1}, 1) \odot (c, e) \\
&= (c\bar{1}, \pi_{\bar{1}}(e)\pi_c(1)) \\
&= (c\bar{1}, \pi_{\bar{1}}(e)) \\
&= (c\bar{1}, e - 1) \\
&\stackrel{R4}{=} (c\bar{1}, (er(e)r(e)) - 1) \\
&\stackrel{N4}{=} (c\bar{1}, (e - 1)r(e)r(e)) \\
&= (c\bar{1}, (e - (ee))r(e)r(e)) \\
&\stackrel{N5}{=} (c\bar{1}, (e - e)r(e)r(e)) \\
&\stackrel{N1}{=} (c\bar{1}, r(e)r(e)r(e)) \\
&\stackrel{R3}{=} (c\bar{1}, r(e)) \\
&= (c\bar{1}, 1).
\end{aligned}$$

Using this we can compute the following:

$$\begin{aligned}
(a, b) - (c, e) &= (a, b) \odot (0 \odot (c, e)) \odot (0 \odot (c, e)) \\
&= (a, b) \odot (c\bar{1}, 1) \odot (c\bar{1}, 1) \\
&= (a(c\bar{1}), \pi_{c\bar{1}}(b)\pi_a(1)) \odot (c\bar{1}, 1) \\
&= (a(c\bar{1}), \pi_{c\bar{1}}(b)) \odot (c\bar{1}, 1) \\
&= (a(c\bar{1})(c\bar{1}), (\pi_{c\bar{1}} \circ \pi_{c\bar{1}})(b) \cdot \pi_{ac\bar{1}}(1)) \\
&= (a, \pi_{c\bar{1}}(b)).
\end{aligned}$$

In particular, if $c \in r(A)$, then $c\bar{1} = \bar{c}$ and $(a, b) - (c, e) = (a, b - c)$. Define

$f : A \mapsto A^0$ by $f(x) = (r(x), d(x))$. For any $x, y \in A$ we have

$$\begin{aligned}
f(xy) &= (r(xy), d(xy)) \\
&\stackrel{R2}{=} (r(x)r(y), d(xy)) \\
&= (r(x)r(y), xy r(xy)) \\
&\stackrel{R7}{=} (r(x)r(y), (xr(x))(yr(y))r(xy)r(xy)) \\
&\stackrel{E10}{=} (r(x)r(y), (xr(x)r(xy)r(xy))(yr(y)r(xy)r(xy))) \\
&\stackrel{E6}{=} (r(x)r(y), (xr(x)r(x)r(x)r(xy)r(xy))(yr(y)r(y)r(y)r(xy)r(xy))) \\
&\stackrel{E9}{=} (r(x)r(y), (xr(x)r(x)r(x)r(y)r(y))(yr(y)r(y)r(y)r(x)r(x))) \\
&\stackrel{E6}{=} (r(x)r(y), (xr(x)r(y)r(y))(yr(y)r(x)r(x))) \\
&= (r(x)r(y), (d(x)r(y)r(y))(d(y)r(x)r(x))) \\
&= (r(x), d(x)) \odot (r(y), d(y)) \\
&= f(x) \odot f(y).
\end{aligned}$$

Moreover

$$\begin{aligned}
f(x - y) &= (r(x - y), d(x - y)) \\
&= (r(x - y), (x - y)r(x - y)) \\
&\stackrel{N9}{=} (r(x), (x - y)r(x)) \\
&\stackrel{N3}{=} (r(x), ((xr(x)) - r(y))r(x)r(x)) \\
&\stackrel{N4}{=} (r(x), ((xr(x)r(x)r(x)) - r(y))) \\
&\stackrel{E6}{=} (r(x), ((xr(x)) - r(y))) \\
&= (r(x), d(x) - r(y)) \\
&= (r(x), d(x)) - (r(y), d(y)) \\
&= f(x) - f(y).
\end{aligned}$$

The above calculations show that f is a BEAN homomorphism. We defined A^0 in a way that $f(A) \subset A^0$, so $\mathbf{A} \leq \mathbf{A}^0$ is a subreduct of \mathbf{H} . \square

This finishes what we needed to know about BEANs, now we will apply their properties to the two remaining mixed cases.

4.3 BEANs with partial semilattice

There are only two cases remaining, the $(\leftrightarrow, -, \widehat{\wedge})$ -subreducts and the $(\leftrightarrow, -, \sqcap)$ -subreducts. We will avoid repeating too much, as we only need to combine some of the previous approaches. The main idea is again to get rid of the operation $-$ by enlarging the universe. The proofs of congruence orderability apply the fact that each of $\widehat{\wedge}, \sqcap, -$ provides, in general, a stronger order than \cdot . However, $\widehat{\wedge}$ only orders the regular part, \sqcap only the dense part and $-$ induces a weaker ordering than the other two.

Definition 4.20. Let $\mathcal{V}_{EARS+BEAN}$ be the class of such algebras $\mathbf{A} = (A, \cdot, \widehat{\wedge}, -)$, that $(A, \cdot, \widehat{\wedge})$ is an EARS, $(A, \cdot, -)$ is a BEAN and the two identities $x\widehat{\wedge}x \approx x - 1, (z - (x\widehat{\wedge}y))r(x)r(x) \approx (z - y)r(x)r(x)$ hold. We can use either $x\widehat{\wedge}x$ or $x - 1$ as $r(x)$ due to the first identity.

The first added identity ensures that the regularization defined using both operations is the same. The second is the counterpart of M5 from the definition of EARS.

Lemma 4.21. Let $\mathbf{H} = (H, \wedge, \vee, \rightarrow, 1, 0)$ be a Heyting algebra. Define $x\widehat{\wedge}y = r_{\mathcal{H}}(x) \wedge r_{\mathcal{H}}(y)$, $x - y = (x \leftrightarrow (\neg y)) \leftrightarrow (\neg y)$. The $(\leftrightarrow, \widehat{\wedge}, -)$ -reduct of \mathbf{H} is in the class $\mathcal{V}_{EARS+BEAN}$.

Proof. We only need to verify that the two identities from the definition are satisfied in \mathbf{H} . Using the language of Heyting algebras they can be written as

$$(x00) \wedge (x00) \approx x00,$$

$$z((x\widehat{\wedge}y)0)((x\widehat{\wedge}y)0)(x00)(x00) \approx z(y0)(y0)(x00)(x00).$$

The first identity is true by the idempotence of \wedge . To show the second one, we recall that the identities from the definition of EARS are true in \mathcal{H} . Hence,

$$\begin{aligned} (y0)\widehat{\wedge}x &\stackrel{M4}{=} (y\widehat{\wedge}x)(0\widehat{\wedge}x)(x00) \\ &\stackrel{S1}{=} (x\widehat{\wedge}y\widehat{\wedge}x)(0\widehat{\wedge}x)(x00) \\ &\stackrel{M4}{=} ((x\widehat{\wedge}y)0)\widehat{\wedge}x \end{aligned}$$

and we apply the above to show

$$\begin{aligned}
z((x\widehat{\wedge}y)0)((x\widehat{\wedge}y)0)(x00)(x00) &\stackrel{M5}{=} z(((x\widehat{\wedge}y)0)\widehat{\wedge}x)((x\widehat{\wedge}y)0)\widehat{\wedge}x \\
&= z((y0)\widehat{\wedge}x)((y0)\widehat{\wedge}x) \\
&\stackrel{M5}{=} z(y0)(y0)(x00)(x00).
\end{aligned}$$

□

Now, we want to reduce the congruence properties to already known ones.

Lemma 4.22. Let $\mathbf{C} = (C, \cdot, \widehat{\wedge}, -)$ be an algebra from $\mathcal{V}_{EARS+BEAN}$. Let \mathbf{A} be its $(\cdot, \widehat{\wedge})$ -reduct and \mathbf{B} be its $(\cdot, -)$ -reduct. For any $c \in C$ we have

$$\Theta_{\mathbf{C}}(1, c) = \Theta_{\mathbf{A}}(1, r(c)) \vee \Theta_{\mathbf{B}}(1, d(c)).$$

Moreover, $T_c = \{(x, y) : x\widehat{\wedge}c = y\widehat{\wedge}c\}$ is a congruence and $\Theta_{\mathbf{C}}(1, c) \subset T_c$.

Proof. As usual, because $c = d(c)r(c)$, we have

$$\Theta_{\mathbf{C}}(1, c) = \Theta_{\mathbf{C}}(1, r(c)) \vee \Theta_{\mathbf{C}}(1, d(c)).$$

Let $a = r(c)$ and take $\varphi = \Theta_{\mathbf{A}}(1, a)$. As it is EARS congruence, by Lemma 2.8 (b), we have

$$\varphi = \{(x, y) : x\widehat{\wedge}a = y\widehat{\wedge}a, d(x)aa = d(y)aa\}.$$

Take any $(x, y) \in \varphi, z \in C$. Then

$$\begin{aligned}
(x - z)\widehat{\wedge}a &= r(x - z)\widehat{\wedge}a \\
&\stackrel{N9}{=} r(x)\widehat{\wedge}a \\
&= r(y)\widehat{\wedge}a \\
&\stackrel{N9}{=} (y - z)\widehat{\wedge}a.
\end{aligned}$$

Moreover

$$\begin{aligned}
d(x-z)aa &= (x-z)r(x-z)aa \\
&\stackrel{N3}{=} (d(x)-z)r(x)r(x-z)aa \\
&\stackrel{N9}{=} (d(x)-z)r(x)r(x)aa \\
&\stackrel{N4}{=} ((d(x)r(x)r(x))-z)aa \\
&\stackrel{R4}{=} (d(x)-z)aa \\
&\stackrel{N4}{=} (d(x)aa) - z,
\end{aligned}$$

and similarly $d(y-z)aa = (d(y)aa) - z$, which shows that $(x-z, y-z) \in \varphi$. On the other hand,

$$(z-x)\widehat{\wedge}a \stackrel{N9}{=} r(z)\widehat{\wedge}a \stackrel{N9}{=} (z-y)\widehat{\wedge}a.$$

Using the second identity that we added to $\mathcal{V}_{EARS+BEAN}$ we have

$$\begin{aligned}
d(z-x)aa &= (z-x)r(z-x)aa \\
&\stackrel{E10}{=} (z-x)aa(r(z-x)aa) \\
&\stackrel{R3}{=} (z-x)aar(z-x) \\
&\stackrel{N9}{=} (z-x)aar(z) \\
&= (z-(x\widehat{\wedge}a))aar(z),
\end{aligned}$$

and similarly $d(z-y)aa = (z-(y\widehat{\wedge}a))aar(z)$ so $(z-x, z-y) \in \varphi$. This shows that φ is still a congruence in the larger language, therefore it must be the minimal congruence, so $\Theta_{\mathbf{C}}(1, r(c)) = \Theta_{\mathbf{A}}(1, r(c))$.

Notice that the same calculations show that $T_c = \{(x, y) : x\widehat{\wedge}c = y\widehat{\wedge}c\}$ is closed under $-$. We have shown in Lemma 2.8 (a), that T_c is an EARS congruence. So it is an EARS+BEAN congruence and obviously $\Theta_{\mathbf{C}}(1, r(c)) \subset T_c$.

Let $b = d(c)$ and $\psi = \Theta_{\mathbf{B}}(1, b)$. Consider the equivalence relation $\ker r = \{(x, y) : r(x) = r(y)\}$, it is closed under \cdot and, by N9, also under $-$. Moreover, $(1, b) \in \ker r$, which means that $\psi \subset \ker r$. If we take any $x, y, z \in A$, $(x, y) \in \psi$, then $r(x) = r(y)$ and $x\widehat{\wedge}z \stackrel{S1}{=} r(x)\widehat{\wedge}z = r(y)\widehat{\wedge}z \stackrel{S1}{=} y\widehat{\wedge}z$. Therefore, ψ is closed under $\widehat{\wedge}$, so it is a congruence and $\Theta_{\mathbf{C}}(1, d(c)) = \psi$. Now notice $\Theta_{\mathbf{C}}(1, d(c)) \subset \ker r \subset T_c$ for any c , so $\Theta_{\mathbf{C}}(1, c) \subset T_c$. \square

Theorem 4.23. *The variety $\mathcal{V}_{EARS+BEAN}$ coincides with the class of $(\cdot, \widehat{\wedge}, -)$ -subreducts of Heyting algebras.*

Proof. Let $\mathbf{A} = (A, \cdot, \widehat{\wedge}, -)$ be any algebra in the class $\mathcal{V}_{EARS+BEAN}$. To show congruence orderability, we can simply repeat the proof of Theorem 2.9. The only difference is that we use BEAN reducts and their orderability instead of equivalential algebras. The above lemma serves the same function as Lemma 2.8 in EARS. If we have two elements $a, b \in A$ and they generate the same principal congruence, then $(1, b) \in T_a$ and $(1, a) \in T_b$, which implies $r(a) = r(b)$. Then we take the $(\cdot, -)$ -reduct of \mathbf{A} , which is a BEAN. It is orderable; and just like in the proof for EARS, we obtain $(d(a), d(b)) \in \Theta_{\mathbf{A}}(1, r(a))$. Finally, we use the formula for $\Theta_{\mathbf{A}}(1, r(a))$ (which is the same as in EARS) to show $a = b$.

To verify that EARS+BEAN algebras are exactly the subreducts of Heyting algebras, we show that they are subreducts of EARS with zero using a similar construction as for BEANs. We define $\mathbf{A}^0 = (A^0, \odot, \widehat{\wedge}, 0)$, where $A^0, \odot, 0$ are as before. For any $(a, b), (c, e) \in A^0$ we define

$$(a, b) \widehat{\wedge} (c, e) = \begin{cases} (a \widehat{\wedge} c, 1) & \text{if } a, c \in r(A), \\ \left(\overline{(x \widehat{\wedge} c)} c, 1 \right) & \text{if } x, c \in r(A), a = \bar{x}, \\ \left(\overline{(a \widehat{\wedge} y)} a, 1 \right) & \text{if } a, y \in r(A), c = \bar{y}, \\ \left(\overline{(x \widehat{\wedge} y)} xy, 1 \right) & \text{if } x, y \in r(A), a = \bar{x}, c = \bar{y}. \end{cases}$$

The formulas for the three later cases are the result of rewriting $\neg x \wedge c, a \wedge \neg y, \neg x \wedge \neg y$ in a Boolean algebra. Alternatively, one can also look at this definition as the only way to extend $\widehat{\wedge}$ so that the identity M4 holds and $\bar{a} = a\bar{1}$.

Now, we need to show that \mathbf{A}^0 is an EARS with zero and $\mathbf{A} < \mathbf{A}^0$. In fact, as we already know \mathbf{A}^0 is an equivalential algebra with zero. We only need to verify S1, S2, S3, M4, M5 and that zero agrees with the EARS part ($x00 = x \widehat{\wedge} x, x \widehat{\wedge} 0 = 0$). We skip the proofs of the semilattice properties and M4 because those regard only regular elements of \mathbf{A}^0 , which again is just a Boolean group $r(A)$ extended by one dimension. We check from the definition that the identities $(a, b) \widehat{\wedge} (a, b) = (a, 1)$ and $(a, b) \widehat{\wedge} 0 = (a, b) \widehat{\wedge} (\bar{1}, 1) = (\bar{1}, 1)$ hold regardless of whether $a \in r(A)$ or $a \in$

$R(A^0) \setminus r(A)$, so the structures imposed by zero and EARS agree as intended. To check M5, we recall from the proof of Lemma 4.18, that we can write

$$(a, b) \odot (c, e) \odot (c, e) = (a, \pi_c(b)\pi_a(e)\pi_a(e)).$$

If we assume that we work with regular elements, this can be simplified further to

$$(a, b) \odot r((c, e)) \odot r((c, e)) = (a, b) \odot (c, 1) \odot (c, 1) = (a, \pi_c(b)).$$

This implies that M5 holds in \mathbf{A}^0 if and only if for any $a, c \in R(A^0)$ we have $\pi_{a\hat{\wedge}c} = \pi_a \circ \pi_c$. In the definition of EARS+BEAN we assumed the identity $(z - (x\hat{\wedge}y))r(x)r(x) \approx (z - y)r(x)r(x)$, which is the same as saying that

$$x, y \in r(A) \Rightarrow \pi_{(\overline{x\hat{\wedge}y})y} = \pi_{\bar{x}} \circ \pi_y.$$

Using this fact and the definition of $\hat{\wedge}$ we prove M5 by splitting it into four cases:

- If $a, c \in r(A)$, then $\pi_{a\hat{\wedge}c} = \pi_a \circ \pi_c$ follows from direct application of M5 in \mathbf{A} ;
- If $x, c \in r(A)$, $a = \bar{x}$, then $\pi_{a\hat{\wedge}c} = \pi_{(\overline{x\hat{\wedge}c})c} = \pi_{(\overline{x\hat{\wedge}c})c} = \pi_{\bar{x}} \circ \pi_c = \pi_a \circ \pi_c$.
- If $a, y \in r(A)$, $c = \bar{y}$, then $\pi_{a\hat{\wedge}c} = \pi_{(\overline{a\hat{\wedge}y})a} = \pi_{(\overline{a\hat{\wedge}y})a} = \pi_{\bar{y}} \circ \pi_a = \pi_a \circ \pi_c$.
- If $x, y \in r(A)$, $a = \bar{x}$, $c = \bar{y}$, then $\pi_{a\hat{\wedge}c} = \pi_{(\overline{x\hat{\wedge}y})xy} = \pi_{(\overline{x\hat{\wedge}y})xy} = \pi_{(\overline{x\hat{\wedge}(xy)})} = \pi_{\bar{x}} \circ \pi_{xy} = \pi_{\bar{x}} \circ \pi_{(\bar{x})(\bar{y})} = \pi_{\bar{x}} \circ \pi_{\bar{y}} = \pi_a \circ \pi_c$.

Finally, we check that the function $f : A \ni x \mapsto (r(x), d(x)) \in A^0$ preserves $\hat{\wedge}$ as

$$f(x\hat{\wedge}y) = (r(x\hat{\wedge}y), 1) = (r(x)\hat{\wedge}r(y), 1) = (r(x), 1)\hat{\wedge}(r(y), 1) = f(x)\hat{\wedge}f(y).$$

Each EARS+BEAN algebra is a subreduct of an equivalential algebra with zero (with $0 \neq 1$), so it is also a subreduct of a Heyting algebra. \square

The last case connects the structures of BEANs and EADS.

Definition 4.24. Let $\mathcal{V}_{EADS+BEAN}$ be the class of such algebras $\mathbf{A} = (A, \cdot, \sqcap, -)$, that (A, \cdot, \sqcap) is an EADS, $(A, \cdot, -)$ is a BEAN, and the two identities $x(x\sqcap x) \approx x - 1$, $(x - y)\sqcap x \approx x\sqcap x$ hold.

Again, the first new identity ensures that the regularizations are the same. The second identity is a counterpart of the M'3 identity from the definition of EADS $(x \sqcap (xyy) = x \sqcap x)$.

Theorem 4.25. *The variety $\mathcal{V}_{EADS+BEAN}$ coincides with the class of $(\cdot, \sqcap, -)$ -subreducts of Heyting algebras.*

Proof. Both required identities hold in Heyting algebras, the first one is just $x \leftrightarrow d_{\mathcal{H}}(x) \approx r_{\mathcal{H}}(x)$, and the second one follows from M'3 by substituting $\neg y$ in place of y .

Let $\mathbf{A} = (A, \cdot, \sqcap, -)$ be an arbitrary algebra from the class $\mathcal{V}_{EADS+BEAN}$ and \mathbf{B} be its (\cdot, \sqcap) -reduct. For a dense element $x \in d(A)$, recall from Lemma 2.23 (c) that $\Theta_{\mathbf{B}}(1, x) = \theta = \{(y, z) : y \sqcap x = z \sqcap x, r(y) = r(z)\}$. We can rewrite it as a filter: $(1, y) \in \theta$ if and only if $y \sqcap x = x, r(y) = 1$. If we take $(1, y) \in \theta$, then for an arbitrary $z \in A$ we have $r(y - z) \stackrel{N9}{=} r(y) = 1$, and by the second identity from the definition $(y - z) \sqcap x = (y - z) \sqcap (y \sqcap x) \stackrel{S'3}{=} ((y - z) \sqcap y) \sqcap x = d(y) \sqcap x \stackrel{S'1}{=} y \sqcap x = x$. This shows that $1/\theta$ is a BEAN filter, so θ preserves $-$.

Similarly if $x \in r(A)$ and $\varphi = \Theta_{\mathbf{B}}(1, x)$, then from Lemma 2.23 (e), $(1, y) \in \varphi$ if and only if $d(y)r(x)r(x) = 1$ and $r(y) \in \{1, r(x)\}$. If $(1, y) \in \varphi$ and $z \in A$, then $r(y - z) \stackrel{N9}{=} r(y) \in \{1, r(x)\}$ and

$$\begin{aligned}
d(y - z)r(x)r(x) &\stackrel{U'2}{=} r(y - z)(y - z)r(x)r(x) \\
&\stackrel{N3}{=} (d(y) - r(z))r(y)r(y)r(x)r(x) \\
&\stackrel{E8}{=} (d(y) - r(z))r(x)r(x)r(y)r(y) \\
&\stackrel{N4}{=} ((d(y)r(x)r(x)) - r(z))r(y)r(y) \\
&= (1 - r(z))r(y)r(y) \\
&\stackrel{N4}{=} (r(y)r(y)) - r(z) \\
&\stackrel{N2}{=} (r(y) - r(z))(r(y) - r(z)) \\
&= 1.
\end{aligned}$$

Hence, φ preserves $-$, so it is a EADS+BEAN congruence. As usual, we have $\Theta_{\mathbf{A}}(1, x) = \Theta_{\mathbf{A}}(1, r(x)) \vee \Theta_{\mathbf{A}}(1, d(x))$. We know that the two congruences on the right side are the same as on \mathbf{B} . But the \vee operation is defined on the lattice of all

equivalence relations on the universe A , so it is the same whether we calculate it in \mathbf{A} or in \mathbf{B} . This shows that for any $x \in A$, the equality $\Theta_{\mathbf{A}}(1, x) = \Theta_{\mathbf{B}}(1, x)$ holds and congruence orderability follows from congruence orderability of EADS.

To show that any $\mathbf{A} \in \mathcal{V}_{EADS+BEAN}$ is a subreduct, we repeat the construction of \mathbf{A}^0 from BEANs. Moreover, we define \sqcap on the larger set by taking

$$(a, b) \sqcap (c, e) = (1, b \sqcap e).$$

It remains to check that such an object is EADS with zero. Identities E1-E3, R1-R4, S'1-S'4, M'1 and M'2 are guaranteed either by \mathbf{A}^0 being an equivalential algebra with zero or by direct application of identities holding on \mathbf{A} . Now, we check that zero agrees with EADS structure:

$$\begin{aligned} (a, b) \odot 0 \odot 0 \odot (a, b) &= (\bar{a}, b - 1) \odot 0 \odot (a, b) \\ &= (\bar{a}, r(b)) \odot 0 \odot (a, b) \\ &= (\bar{a}, 1) \odot 0 \odot (a, b) \\ &= (a, 1) \odot (a, b) \\ &= (1, b) \\ &= (1, b) \sqcap (1, b). \end{aligned}$$

To show M'3 we observe that for $b \in d(A)$ we have $b \sqcap \pi_c(b) = b$ either by M'3 in \mathbf{A} or by the second identity from the definition of EADS+BEAN. Then, we can check that

$$\begin{aligned} (a, b) \sqcap ((a, b) \odot (c, e) \odot (c, e)) &= (a, b) \sqcap (a, \pi_c(b) \pi_a(e) \pi_a(e)) \\ &= (1, b \sqcap (\pi_c(b) \pi_a(e) \pi_a(e))) \\ &= (1, (b \sqcap \pi_c(b)) \sqcap (\pi_c(b) \pi_a(e) \pi_a(e))) \\ &= (1, b \sqcap (\pi_c(b) \sqcap (\pi_c(b) \pi_a(e) \pi_a(e)))) \\ &\stackrel{M'3}{=} (1, b \sqcap \pi_c(b)) \\ &= (1, b) \\ &= d((a, b)). \end{aligned}$$

The map $x \mapsto (r(x), d(x))$ is a homomorphism with respect to \sqcap , so \mathbf{A} is a subreduct of \mathbf{A}^0 . Because \mathbf{A}^0 is an EADS algebra with zero in which $0 \neq 1$, it is a

$(\leftrightarrow, \sqcap, 0)$ -subreduct of a Heyting algebra. Therefore, \mathbf{A} is a $(\leftrightarrow, \sqcap, -)$ -subreduct of a Heyting algebra. \square

This is where our research ends for now. We have previously shown that under assumptions of Theorem 3.1 there are six possible classes of subreducts that contain mixed type algebras. Now we know, that five of them are varieties, and the last one is a quasivariety. In each case mixed type arises, because the algebras behave differently on their dense and regular parts.

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