dr hab. inż. Anna Zamojska-Dzienio
Wydział Matematyki i Nauk Informacyjnych
Politechnika Warszawska
ul. Koszykowa 75
00-662 Warszawa
anna.zamojska@pw.edu.pl

Warszawa, July 30, 2025

Review of the doctoral dissertation of Eryk Lipka, M.Sc., entitled *Fregean fragments of Intuitionistic Propositional Calculus with mixed congruence type*

It is well-known that Heyting algebras constitute an algebraic semantics for Intuitionistic Propositional Calculus. Varieties of algebras generated by reducts of Heyting algebras with equivalence: $x \leftrightarrow y := (x \to y) \wedge (y \to x)$ contained in the reduced language provide a rich class of examples of congruence permutable Fregean varieties. In the thesis under review the author focuses on the case when algebras in such varieties are of mixed type, i.e., between type **2** and type **3** (as understood in Tame Congruence Theory), which means that some parts of the algebra behave as being locally solvable (type **2**) and others as being congruence distributive (type **3**).

Namely, the author is looking for classes of subreducts of Heyting algebras which are congruence permutable Fregean (quasi)varieties containing mixed type algebras. The *big* research question is: *assuming we have a congruence permutable Fregean class of subreducts of Heyting algebras, when this class is of mixed type?*

This thesis is a starting point of this new direction of research and hence gives a partial answer under additional requirements on the language:

(1) The equivalence operation $\leftrightarrow$ must be included;
(2) Only terms constructed by means of operations from the set $\{\to, \wedge, 0\}$, i.e., terms that can be written without the $\vee$ operation, can be used;
(3) Only *binary* terms are considered.

The reasons for these assumptions are natural and well justified in the thesis, e.g., a congruence permutable Fregean variety always has a term acting as equivalence. The research interests restrict only to the algebraic part, although it is mentioned that some of the results might be rephrased in logic terms. The main results are that there are exactly six classes of such subreducts: five of them are varieties and one of them is a quasivariety. A finite basis of axioms is provided in each case. The algebras in the classes considered behave differently on the part consisting of regular elements and on the part consisting of dense elements. This is why one obtains a mixed type here.

Hence, the dissertation provides a complete picture in this particular case. The idea behind the proof is natural and typical (this does not mean *easy*!) in this kind of research. First, one should "guess" (invent) a variety that can contain subreducts in which one is interested, then demonstrate some properties of this variety which will be useful for the aims, and finally show that the variety is actually equal to the class of the considered subreducts.

The thesis has a classical construction and consists of an introduction, four chapters and a bibliography list. The lack of a table of contents is very inconvenient for the reader.

The Introduction plays its role, briefly explaining the motivation behind this research and describing the content of the thesis. It ends with a discussion of possible directions of further research. However, one could create a final chapter about this, collecting other fragments on the topic scattered throughout the thesis. It would provide suitable closure, especially since there are still many open questions and problems that can be investigated in the future.

Chapter 1 provides necessary background on universal algebra, Fregean algebras, and Fregean varieties and their properties. It is well written, containing many comments and remarks which show good knowledge and understanding of the research area by the author.

Chapter 2 presents the first two classes of subreducts of Heyting algebras that contain algebras of mixed type. The goal was to extend Przybylo's previous results on a variety generated by a single reduct of a three-element Heyting algebra. The first part of the chapter (Section 2.1) investigates subreducts with respect to the set $\mathcal{G} = \{\leftrightarrow, \hat{\wedge}\}$ with $x \leftrightarrow y := (x \to y) \wedge (y \to x)$ and $x \hat{\wedge} y := (\neg\neg x) \wedge (\neg\neg y)$. Here, the operation $\hat{\wedge}$ turns out to be a partial semilattice operation restricted to regular elements. The main result is Theorem 2.7 which states that the class of all $\mathcal{G}$-subreducts of Heyting algebras is a variety. Moreover, a finite axiomatization is provided (Definition 2.1), and basic properties of the variety are described, e.g., it is congruence permutable, Fregean, locally finite, primitive, all nontrivial algebras have unique extensions, etc. I would like to mention that the symbol $\mathcal{H}_{emd}$ used in this part, was not introduced before. This part is based on a joint paper co-authored with the supervisor. The second part (Section 2.2) investigates in a similar way the class of $\mathcal{H}$-subreducts with $\mathcal{H} = \{\leftrightarrow, \Cap\}$, where $x \Cap y := (\neg\neg x \to x) \wedge (\neg\neg y \to y)$. Now, the operation $\Cap$ is a semilattice operation on dense elements. The procedure is analogous to that for $\mathcal{G}$-subreducts. The class of all $\mathcal{H}$-subreducts of Heyting algebras is a variety (Theorem 2.28) with a finite equational basis (provided in Definition 2.16).

Chapter 3 focuses on explaining the method used in the thesis for searching other classes of subreducts of Heyting algebras that will be congruence permutable, Fregean and of mixed type. The authors restricts his investigations to Brouwerian semilattices with zero, i.e., $(\wedge, \to, 0, 1)$-subreducts of Heyting algebras, which also form a variety. The universal algebraic considerations, together with restrictions on the language and family of terms assumed here, enables the use of computer assistance to perform initial investigations. The first result is the full description of the sublattice of clones on the free Brouwerian semilattice over two generators which contain equivalence and regularization (summarized in Figure 3.1). Later, in Theorem 3.19, it is shown that the only possibility to obtain an algebra of a mixed type as a subreduct is to have a regularization in the clone of operations (generated in this case by the two-element set consisting of equivalence and an additional binary term). Together with Theorem 3.16, it gives six classes of subreducts of Brouwerian semilattices with zero, which satisfy the requirements imposed by the author. Two of them have already been described in Chapter 2. The main result of the thesis is formulated here as Theorem 3.1 at the beginning of the chapter, however in a way not revealing its full content. I think that it could be recalled later in a more precise and detailed way. In fact, the thesis is devoted to the proof of Theorem 3.1.

Chapter 4 studies the four remaining classes. Section 4.1 focuses on $(\leftrightarrow, \hat{\wedge}, \neg)$- and $(\leftrightarrow, \Cap, \neg)$-subreducts. It is shown that the first class coincides with the variety of equivalential algebras with a semilattice operation on regular elements and with zero (Theorem 4.4) and the latter with the quasivariety of separating (they separate 0 and 1) equivalential algebras with a semilattice operation on dense elements and with zero (Theorem 4.6). In Section 4.2 it is shown that $(\leftrightarrow, -)$-subreducts of Heyting algebras, with $x - y := x(\neg y)(\neg y)$, form a variety. Since this variety behave differently than previous ones due to the properties of a term $-$, e.g., algebras here do not have unique extensions, the proofs in this part are more demanding. Finally, Section 4.3 describes $(\leftrightarrow, \hat{\wedge}, -)$- and $(\leftrightarrow, \Cap, -)$-subreducts. Theorems 4.23 and 4.25 show that in each case we obtain the variety of algebras with a finite equational basis.

The dissertation is well written. There is clearly stated mathematical problem, and it is completely solved in some particular, but natural, cases. I appreciate the additional comments and explanations provided generously by the author throughout the thesis which show his deep knowledge and understanding of his field of research. Application of computer tools fits in to the current trends of modern universal algebra and shows its usefulness for such research. In this thesis, such an application is always followed by the detailed algebraic analysis which allows one to eliminate many cases before starting to use “brute force”. The results obtained are new, interesting, and encourage further research in this direction, to explore the frontiers. The author himself indicates many such areas here. In my opinion, the thesis shows that the author possesses the necessary background and skills to conduct further research on his own.

Also, on the editorial side, the thesis is quite well written, although some errors appeared. Since they are of minor importance, they do not affect my positive opinion of the thesis. I will mention some of them.

- title - there is *conguence* instead of *congruence*;
- page 2, lines 13-14, there is *Any finite congruence algebra* instead of *Any finite congruence permutable algebra*;
- page 3, line 15, two typos: *opeartions* and *cojunction*;
- page 4, line 5, there should be *contains* (it refers to *Language*);
- Definition 1.2. starts with *An Algebra* instead of *An algebra*;
- Definition 1.24., there is a phrase *with two binary constants* in the first line, probably this should be *with two binary operations and two constants*;
- page 16, line -5, there is unnecessary *use* in front of *introduce*;

- Lemma 1.48., there is an unnecessary bracket in $(a, b \in A$ (second line). In Proof instead of writing *from the previous Theorem* one should use its name: Correspondence Theorem which was introduced a few lines above;
- page 21, line -5, instead of *This paper* it would be better to use *This thesis*;
- Lemma 1.76 is rather a remark;
- caption of Figure 1.2, a typo *completly*;
- page 32, line -5, instead of *it is two-element.* it should be *it has two elements* or *it is a two-element algebra*;
- title of Chapter 2, I would write "Equivalential algebras with *a* partial semilattice" and I would add *structure* or *operation*; similarly in the first line of Definition 2.1: *a* semilattice of regular elements, etc.; similar comments apply to Section 4.3
- page 39, line 15, a typo *subducts*;
- page 61, line 4, instead of *algebras* it should be *varieties* since we describe *classes*;
- page 62, line 4, there should be $\{\to, \wedge, 0\}$ (an incorrect bracket was used); in general at this page the choice of brackets looks random: { } vs. ( ) for *pairs* - I cannot see any rule behind this;
- Theorem 3.1., the end of the first line - there is unnecessary *a* unless there should be some noun, e.g., *class of algebras*;
- page 67, line -8, I think *distinguish* suits better than *differentiate* in this context;
- page 86, line 15, a typo *semmilattice*;
- Lemma 4.2., there should be *hence* $\mathcal{E}_0$ *is* in the second line.

In conclusion, I declare that the reviewed doctoral dissertation fulfills both the statutory and customary requirements for a doctoral thesis in mathematics and I request that Eryk Lipka should be admitted to the further stages of the doctoral process.

Anna Zamojska-Dzienio