REPORT ON ERYK LIPKA'S PHD DISSERTATION *FREGEAN FRAGMENTS OF INTUITIONISTIC PROPOSITIONAL CALCULUS WITH MIXED CONGRUENCE TYPE*

1. Introduction

Two crowning jewels of contemporary universal algebra are commutator theory and tame congruence theory. Their applicability in algebraic logic is unfortunately quite limited because a typical variety of logic such as Boolean algebras, Heyting algebras, modal algebras or residuated lattices is congruence distributive so the commutator of two congruences is just their set-theoretical intersection and hence commutator theory trivialises. Moreover, tame congruence theory directly applies only to locally finite varieties and, except Boolean algebras, typical varieties of logic are not locally finite.

A very natural variety of logic to which both commutator theory and tame congruence theory nontrivially apply is the variety of equivalential algebras. It is locally finite and congruence permutable (so tame congruence theory applies), but not congruence distributive (so commutator theory is nontrivial). Equivalential algebras are moreover Fregean, which gives a very pleasant description of subdirectly irreducibles, as algebras with a unique subcover of the top/unit element.

It is thus very natural to investigate other classes of algebras of logic to which commutator theory and tame congruence theory apply nontrivially, and a natural starting point for this investigation is to study certain expansions of equivalential algebras, or viewed from a different perspective, certain subreducts of Heyting algebras (or Brouwerian semilattices). This is precisely the starting point for Eryk Lipka's dissertation.

2. The candidate

Eryk Lipka earned the title of *magister matematyki* in 2017 at the Jagiellonian University. He has never previously been a PhD candidate.

He has so far published 4 articles in respectable journals. Remarkably, the articles cover quite a range of topics: from automatic structures, through combinatorics, to algebraic logic. Of these, combinatorics and algebraic logic are well represented in the dissertation. Also present in the background are Lipka's programming skills: many results in the thesis are obtained by formulating computer assisted conjectures, and then proving (or re-proving) them by hand.

3. Layout and content of the dissertation

The dissertation consists of 4 chapters preceded by a short introduction. Chapter 1 gives the necessary background on universal algebra, Fregean varieties and equivalential algebras. Chapter 2 contains results on two particular varieties of subreducts of Heyting algebras, called EARS and EADS. The results on EARS have already been published, the results on EADS are obtained by applying the methods developed for EARS. Chapters 3 and 4 extend these results further, to obtain a full classification of certain special subreducts of Brouwerian semilattices with zero, namely, subreducts that are congruence permutable, Fregean and of mixed type (in the tame congruence theory sense). The main theorem of the dissertation in Theorem 3.1, stating that there are precisely six such classes of subreducts; five of them are varieties and one is a quasivariety; all are finitely based.

The results are obtained by analysing the (dual representation of the) free Brouwerian semilattice on 2 generators. Chapter 3 analyses the subreducts without negation, Chapter 4 analyses the subreducts with negation. The technique employed is combinatorics on top of duality for the free Brouwerian semilattices. For finding equational and quasiequational bases a nice trick of "non-existence of a minimal counterexample" is used. The main result is easy to state, but requires an extensive knowledge of universal algebra, duality and combinatorics to prove, and remarkable programming skills to obtain the initial data (conjectures, and possibly some proofs). The candidate has chosen to hide these skills behind modest statements such as this, from page 63: "[we] use computer assistance to perform initial investigation".

4. Comments

The dissertation is in general written well, although at places in gives the impression of work done under the pressure of time. For example, a number of times we read "in this paper", where we should read "in this dissertation/thesis": clearly a copy-and-paste error. Copying and pasting fragments of one's previous work is common, but a second reading would help avoid such a giveaway. There are quite a few typos, and several awkward formulations that could also have been avoided. They are generally minor, so I will only point out three that made me stop for a moment to catch the meaning.

- p.2, l.-13: congruence algebra $\rightarrow$ congruence permutable algebra
- p.31, second displayed equation: the prime in M' clashes with the prime on the right-hand side standing for complementation
- p.31, l.-12: the condition (†) $\rightarrow$ the condition (†) of Theorem 1.65

The notation is by and large standard, and this includes using $*$ as the second largest element of the algebra. It is indeed traditional, but I have heard a number of people complain about this tradition, pointing out that $*$ suggests a binary operation to the uninitiated. I have nothing against $*$ in the dissertation, but I would discourage using it in journal submissions.

The crucial operation of equivalence is at times written in the "logical" fashion, as $\leftrightarrow$, sometimes "algebraically" as $\cdot$ or as juxtaposition. Again, in general this does not cause problems for the reader familiar with equivalential algebras, but for other readers it may be confusing. If a journal article is planned, based on the dissertation—and I certainly encourage that—I would either avoid $\leftrightarrow$ altogether or explain why different symbols are used.

The symbol r for regularisation is used a little sloppily: at times it is a primitive operation (Section 1.6), at other times it is a defined operation (Definition 2.1). In Example 1.87 and later we also see $r_{\mathcal{H}}$ which stands for the double negation operation $(x \rightarrow 0) \rightarrow 0$ in Heyting algebras. Again, it is clear to readers familiar with Heyting algebras, others may get confused.

My critical remarks above are not intended to carry too much weight. It should be clear from Section 3 that my opinion of Eryk Lipka's work is high. The rest of this section suggests two further slight improvements.

The restriction of the language to binary Heyting algebra terms is well justified in the dissertation, and in fact there is another argument for this choice. Namely, every term-reduct of Heyting algebra language is in fact also a term-reduct or a certain binary fragment of Heyting algebra language, since all fundamental operations of that language are at most binary. As stated in the thesis, a thorough investigation of all term-reducts would be prohibitively complicated, so some restriction is needed, and the restriction to binary is the first one would think of. Since we also want to maintain the property of local finiteness, one may equally well restrict attention to subreducts of Brouwerian semilattices with zero from the very beginning.

Theorem 2.15, although not indicated as one of the main results, is quite interesting, and in fact its conclusion can be strengthened to say "all algebras" instead of "all finitely generated algebras". To be precise, we have the following.

Theorem 1. *Let $\mathcal{V}$ be a variety satisfying the five conditions of Theorem 2.15. Then, every $\mathbf{A} \in \mathcal{V}$ is a subreduct of a Heyting algebra.*

Proof. Since every algebra embeds into an ultraproduct of its finitely generated subalgebras, we have that an arbitrary algebra $\mathbf{A} \in \mathcal{V}$ embeds into an ultraproduct $\prod_{i \in I} \mathbf{A}_i/U$ for some finitely generated algebras $\mathbf{A}_i \in \mathcal{V}$. By local finiteness, each $\mathbf{A}_i$ is finite, hence, by Theorem 2.15, $\mathbf{A}_i$ is a subreduct of a Heyting algebra $\mathbf{B}_i$ for some Heyting algebra $\mathbf{B}_i$. Then $\prod_{i \in I} \mathbf{A}_i/U \leq \prod_{i \in I} \mathbf{B}_i/U$ via the coordinatewise embeddings. Since $\mathbf{A}$ embeds into $\prod_{i \in I} \mathbf{A}_i/U$, we conclude that $\mathbf{A}$ is a subreduct of a Heyting algebra. $\square$

5. Conclusion

Eryk Lipka's dissertation *Fregean fragments of Intuitionistic Propositional Calculus with mixed congruence type* fully satisfies all requirements for a PhD dissertation, and therefore I unreservedly recommend proceeding to the final stage: the defence.

Tomasz Kowalski