Review of the doctoral dissertation

Geometric properties of arrangements of low-degree curves

by **MSc. Łukasz Merta**

prepared for the Mathematics Discipline Council
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Reviewer: Prof. Giuseppe Favacchio

1. General description of the dissertation

The doctoral dissertation of MSc. Łukasz Merta, entitled *Geometric properties of arrangements of low-degree curves*, is devoted to questions in the geometry of plane algebraic curves and their arrangements. The main objects studied in the dissertation are configurations of low-degree curves, special points on Fermat curves, higher order tangencies, bitangent conics, and free arrangements of smooth conics.

The thesis consists of an introduction, a preliminary chapter, three main chapters, and an appendix containing computer code used in the computations. The first main part concerns sextactic points and points of type 9 on the Fermat cubic, together with associated configurations of tangent conics and cubics. The second part studies special tangency phenomena for certain quartic curves, including the Fermat quartic and the Komiya–Kuribayashi quartic. The last main part is devoted to free arrangements of three smooth conics with ADE singularities.

The dissertation lies at the intersection of classical projective geometry, the theory of plane algebraic curves, and computational algebraic geometry. The use of computer algebra, especially the software Singular, plays a significant role throughout the work.

2. Scientific content and positive aspects

The topics considered in the dissertation are interesting and natural within the geometry of plane curves. The study of special points, high-order tangencies, bitangent conics, and free arrangements connects classical questions with modern computational and combinatorial methods.

One positive feature of the dissertation is the explicit nature of the results. The author provides concrete equations, coordinates, and classifications for several families of geometric objects. Such explicit computations are often difficult and can be valuable, especially when they reveal configurations that are not immediately accessible by purely theoretical considerations.

In Chapter 2, the candidate obtains several explicit results on Fermat curves. I mention in particular Proposition 2.1, which gives a formula for the second Hessian of F_n, Proposition 2.2, where the coordinates of the sextactic points are determined, and Proposition 2.5, which describes the points of type 9 on the Fermat cubic as a complete intersection. I also find the method based on Abel's theorem and division polynomials interesting: it gives a conceptual way to relate high-order contact phenomena on the Fermat cubic to torsion points on an elliptic curve, before carrying out the explicit computations. The subsequent Propositions 2.8–2.12 provide enumerative and incidence results for bitangent conics associated with sextactic points and points of type 9. These questions are closely related to classical geometry and to the group law on elliptic curves.

Chapter 3 extends similar questions to quartic curves, especially the Fermat quartic and the Komiya–Kuribayashi quartic. Among the concrete results, I mention Proposition 3.11, which describes the sextactic points of the Komiya–Kuribayashi quartic as a complete intersection, and Proposition 3.15, which gives six conics tangent at pairs of distinguished sextactic points on that quartic.

Chapter 4 is devoted to the classification of free arrangements of three smooth conics with ADE singularities, up to projective equivalence. The author uses numerical criteria for freeness, in particular the Du Plessis–Wall theorem recalled as Theorem 4.1, and then analyzes the possible weak combinatorics and their geometric realizability in Section 4.3. The classification problem for free curve arrangements is a non-trivial one, and the explicit treatment of arrangements with ADE singularities provides useful examples and contributes to the understanding of the relation between singularities, combinatorics, and freeness. I also note positively that, in the course of this classification, the author identifies and corrects an erroneous formula appearing in the previous work [Sar10], see Proposition 4.11. This is a useful contribution, since it clarifies one of the explicit models used in the classification.

Overall, the dissertation contains material of mathematical interest and shows that the candidate is able to work with non-trivial explicit problems in algebraic geometry and computational algebra.

3. Main critical remarks

The main reservations I have about the dissertation concern not so much the choice of topics, which I find interesting, but rather the presentation and the mathematical justification of some results.

A substantial part of the dissertation is computational. This is not in itself a weakness: computational methods are a legitimate and powerful tool in modern algebraic geometry. However, when the correctness of a result depends on a computer algebra computation, the dissertation should make clear what exactly has been computed, why the computation proves the stated result, and how completeness and non-degeneracy have been verified.

This is particularly important for results such as Proposition 2.5, Propositions 2.8–2.12, Proposition 3.15, and the classification results of Section 4.3. These are complete lists, enumerative, or classification statements; therefore, the dissertation should explain more clearly how the underlying computations establish completeness.

In Chapter 4, after Theorem 4.5, the thesis derives 25 possible weak combinatorics from numerical constraints and then reduces the list to 9 cases, of which 3 are excluded by explicit proofs in the text and 6 are shown to be realizable, while the remaining 16 exclusions are deferred to [MZZ25]. Since the chapter aims at a complete classification, it would be preferable to include the essential exclusion arguments in the dissertation itself, at least in summarized form, so that the reader can verify the completeness of the classification without relying on an external preprint.

In several places, results are presented after statements such as “it can be checked” or “all computations were done using SINGULAR”, without a sufficient explanation of the underlying algorithm or of its mathematical correctness. In my opinion, the thesis would be significantly improved by a more systematic discussion of the computational methodology. For each major computational result, the author should indicate the input and output of the computation, the mathematical criterion being tested, the reason why the algorithm produces all the required objects and not only some examples, the way degeneracies and exceptional cases are excluded, and how the computations can be reproduced from the code provided.

I also find the preliminary chapter somewhat insufficient for the later use of certain key tools. For instance, the Hessian curve is an essential object in the study of inflection points and appears naturally in the subsequent chapters; moreover, the second Hessian is used in connection with sextactic points. These notions should be introduced carefully in the preliminary chapter, with precise definitions and a short explanation of their geometric role. A similar, although less severe, issue concerns the notion of freeness. The definition in the preliminary chapter is correct, but rather compressed. Since freeness is the central notion of Chapter 4, it would be helpful to add more motivation explaining why free arrangements are geometrically significant and why their classification is a natural problem.

4. Minor remarks and suggestions

I list here some additional suggestions that may help improve the final version of the dissertation.

- Some typographical and bibliographical issues should be corrected. For instance:
  - on p. 31, in the defining equation of the Komiya–Kuribayashi quartic, the term $x^4 + x^4 + z^4$ should presumably be $x^4 + y^4 + z^4$;
  - on p. 19, in the list of sextactic points, the entry S_{11} appears to have only two homogeneous coordinates, so the formula should be checked;
  - on p. 16, the phrase “As previously stated in [SS24]” should probably be replaced by “As shown in [SS24]”;
  - the reference [TM93] should be checked: the label, authors, year, and arXiv reference appear to be inconsistent.
- The dissertation states that parts of Chapter 2 are based on [MZ25a], parts of Chapter 3 extend [MZ25b], and initial results of Chapter 4 were announced in [MZZ25]. In such cases, the relation between the dissertation and the cited papers should be made more explicit. It should be clear which results are recalled from previously published or announced work, which parts are extended in the thesis, and which contributions are genuinely new in the present dissertation.
- The definition of the Milnor and Tjurina numbers in Definition 4.1 is standard, but the transition from a local affine equation $f \in \mathbb{C}[x, y]$ to singularities of projective curves in $\mathbb{P}^2$ could be made more explicit. Since the dissertation mainly deals with projective curves, the use of coordinates with $p = (0, 0)$ should be clarified.

- In Definition 4.1, after recalling the Milnor and Tjurina numbers, it would be helpful to state immediately that for the ADE singularities considered in the chapter one has $\mu_p = \tau_p$. Although this fact is mentioned later near Table 4.3, recalling it at the point where the invariants are introduced would make the exposition clearer.

- The appendix containing SINGULAR code is a valuable addition. However, its usefulness would increase if the code were more directly linked to the propositions in the main text, for example by indicating which script verifies which statement.

5. Conclusion

I consider that the dissertation addresses interesting and relevant problems in the geometry of plane curves and contains sufficient original scientific material. The candidate demonstrates the ability to obtain non-trivial explicit results concerning special points, higher order tangencies, and arrangements of low-degree curves.

My reservations concern mainly the exposition, the methodological presentation of the computational component, and the need for clearer justification of some algorithms and computations. They do not, however, prevent the dissertation from being admitted to the next stage of the doctoral procedure.

I therefore conclude that the submitted dissertation meets the requirements expected of doctoral dissertations and I recommend admitting MSc. Łukasz Merta to the public defence of his doctoral dissertation.