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Geometric properties of arrangements of low-degree curves

PhD Thesis

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Contents

Chapter 0

Introduction

The study of plane algebraic curves and their configurations has a long and rich tradition in algebraic geometry, beginning with classical incidence theorems such as that of Pappus and evolving through the projective geometric insights of Desargues and Pascal on configurations of lines and conics. This classical theory was further deepened in the nineteenth century by fundamental results such as Bézout's theorem on intersections of plane curves, the Plücker formulas relating degree, singularities, and dual curves, and the Cayley-Bacharach theorem governing constraints imposed by point configurations on families of curves. While originally formulated in purely synthetic or enumerative terms, these ideas now find modern expression through combinatorial structures, algebraic invariants of defining ideals, and the analysis of singularities. See Dolgachev's monograph [Dol12] for excellent introduction to this circle of ideas.

In this thesis we focus on arrangements of plane curves of low degree, namely lines, conics, and cubics. The arrangements under consideration arise either from distinguished geometric constructions or are motivated by combinatorial questions and algebraic properties of the associated defining ideals, such as *freeness*. Conversely, an arrangement of curves naturally determines a configuration of points in the plane, for instance via intersection points or higher order contact loci. This duality between configurations of points and arrangements of curves plays a central role throughout the thesis.

In Chapter 2, we study the Fermat curves $F_n : x^n + y^n + z^n = 0$. We determine the coordinates of the sextactic points and derive equations for the conics that are tangent to F_n at these points with multiplicity 6. We then specialize to the Fermat cubic F_3. where we compute the coordinates of the points of type 9 and the equations of cubics tangent to F_3 at these points with multiplicity 9. Finally, we investigate certain configurations of conics which are tangent to F_3 either at two sextactic points or at two points of type 9, with multiplicity 3. Some results contained in this chapter have been already published [MZ25a].

In Chapter 3, we consider two specific quartic curves: the Fermat quartic F_4 and the Komiya-Kuribayashi quartic. We study the line configurations related to the maximal tangency points and the maximal tangency lines of both quartics. In parallel with the approach developed in Chapter 2, we compute the coordinates of the sextactic points on both quartics and we investigate the arrangements of conics tangent at two of these points, now with multiplicity 4. The results of this chapter are an extended version of the published article [MZ25b].

In Chapter 4, we investigate free arrangements of three smooth conics in $\mathbb{P}^2(\mathbb{C})$. We present a complete classification of all such arrangements with ADE singularities, up to projective equivalence. For each arrangement, we explicitly determine defining equations for the conics and compute the coordinates of their singular points. Initial results of this chapter were announced in [MZZ25]. The chapter is much more detailed and contains some new facts.

Finally, the Appendix contains the source code written for use with Singular [DGPS24]. This software was employed extensively to carry out computations throughout the thesis. For clarity, the code included in the Appendix is organized into separate sections. The complete version of the source code is available at `https://github.com/LukaszM111/Singular_code_phd_thesis`.

Chapter 1

Preliminaries and notation

1.1 The projective plane and plane algebraic curves

Throughout this thesis we work over the field $\mathbb{C}$ of complex numbers. We denote by $\mathbb{P}^2(\mathbb{C})$ the complex projective plane. A point $P \in \mathbb{P}^2(\mathbb{C})$ is represented by homogeneous coordinates

$$P = [x : y : z],$$

where $(x, y, z) \in \mathbb{C}^3 \setminus \{0\}$ and $[x : y : z] = [\lambda x : \lambda y : \lambda z]$ for every $\lambda \in \mathbb{C} \setminus \{0\}$.

The homogeneous coordinate ring of $\mathbb{P}^2(\mathbb{C})$ is

$$S = \mathbb{C}[x, y, z],$$

graded by total degree. For a homogeneous polynomial $f \in S$ of degree $d \geq 1$, we define the associated plane algebraic curve by

$$V(f) = \{[x : y : z] \in \mathbb{P}^2(\mathbb{C}) \mid f(x, y, z) = 0\}.$$

The integer $d = \deg f$ is called the *degree* of C.

Definition 1.1
A plane curve $C = V(f)$ is said to be *irreducible* if f is irreducible in S, and *reduced* if f has no repeated irreducible factors.

Definition 1.2
The point P is called a *singular point* of C if

$$f_x(P) = f_y(P) = f_z(P) = 0.$$

Otherwise P is called a *smooth point* of C. The set of singular points of C is denoted by

$\mathrm{Sing}(C)$.

If P is a smooth point of $C = V(f)$, then the *tangent line* T_PC to C at P is given by the linear equation

$$f_x(P)x + f_y(P)y + f_z(P)z = 0.$$

The projective linear group $\mathrm{PGL}(3, \mathbb{C})$ acts naturally on $\mathbb{P}^2(\mathbb{C})$ and hence on plane curves. Two curves are said to be *projectively equivalent* if they lie in the same orbit under this action.

The material of this section is standard; we refer for instance to [Dol12] for a comprehensive introduction to plane algebraic curves and projective geometry.

1.2 Singularities and local intersection multiplicity

Let $C = V(f) \subset \mathbb{P}^2(\mathbb{C})$ be a plane algebraic curve of degree d and let $P = [x_0 : y_0 : z_0] \in C$.

Definition 1.3
Let $C = V(f)$ and let $P \in C$. The *multiplicity* of C at P, denoted by $\mathrm{mult}_P(C)$, is the order of vanishing of a local equation of C at P.

Multiplicity measures the complexity of the singularity and satisfies $\mathrm{mult}_P(C) = 1$ if and only if P is a smooth point of C.

A fundamental notion in the study of plane curves is that of intersection multiplicity.

Definition 1.4
Let $C = V(f)$ and $D = V(g)$ be two plane curves without a common component and let $P \in C \cap D$. The *intersection multiplicity* of C and D at P is denoted by

$$I_P(C, D).$$

We recall that $I_P(C, D) \geq 1$ whenever $P \in C \cap D$, and $I_P(C, D) = 1$ if and only if the curves meet transversely at P. Higher values of $I_P(C, D)$ correspond to tangency or higher order contact.

In particular, if P is a smooth point of C, then the tangent line T_PC satisfies

$$I_P(C, T_PC) \geq 2,$$

and the inequality is strict precisely when P is an *inflection point* or, more generally, a point of higher order contact.

The global behaviour of intersections is governed by Bézout's theorem: if C and D are curves of degrees d and e without a common component, then

$$\sum_{P \in C \cap D} I_P(C, D) = de.$$

The notions recalled above are classical; for detailed accounts we refer to [Dol12, GH78, Dim87].

1.3 Higher order contact and special points on plane curves

Let $C, D \subset \mathbb{P}^2(\mathbb{C})$ be plane curves and let $P \in C \cap D$. The intersection multiplicity $I_P(C, D)$ provides a natural measure of the order of contact of the curves at P.

Definition 1.5
Let $P \in C \cap D$. The curves C and D are said to have *contact of order* k at P if

$$I_P(C, D) \geq k.$$

If $I_P(C, D) = k$, then C and D are said to have *exact contact of order* k at P.

In particular, if P is a smooth point of C, then the tangent line $T_P C$ has contact of order at least 2 with C at P.

A central role in classical projective geometry is played by osculating curves.

Definition 1.6
Let P be a smooth point of a plane curve C. An *osculating curve* of degree d at P is a curve D of degree d maximizing the intersection multiplicity $I_P(C, D)$ among all curves of degree d passing through P.

The simplest instance is the tangent line, which is the osculating curve of degree 1. Higher degree osculating curves lead to special points on C.

Definition 1.7
Let P be a smooth point of C. The point P is called an *inflection point* if

$$I_P(C, T_P C) \geq 3.$$

Inflection points are classically cut out on C by the Hessian curve $H(C)$ of C (see e.g. [Dol12]).

A classical observation due to Cayley asserts that conics provide the next level of approximation of a smooth plane curve after the tangent line.

Proposition 1.1
Let $C = V(f)$ be a smooth plane curve and let $P \in C$. Then there exists a conic Q such that

$$I_P(C, Q) \geq 5.$$

Moreover, for a general point $P \in C$ this intersection multiplicity is equal to 5.

A conic satisfying $I_P(C, Q) \geq 5$ is called the *osculating conic* to C at P.

We briefly indicate how such a conic can be computed. Choose affine coordinates so that $P = (0, 0)$ and the tangent line to C at P is given by $y = 0$. Then locally C can be written in the form

$$y = a_2x^2 + a_3x^3 + a_4x^4 + \cdots .$$

A general affine conic through P with tangent line $y = 0$ has the local parametrization

$$y = \alpha x^2 + \beta x^3 + \gamma x^4.$$

Matching the coefficients up to order four determines uniquely the conic having maximal contact with C at P, which yields contact of order at least 5.

An equivalent coordinate–free description can be given in homogeneous coordinates. Let $P = [x_0 : y_0 : z_0]$ be a smooth point of $C = V(f)$. Denote by $H_f(P)$ the Hessian matrix of f at P. Then the osculating conic at P is obtained by combining the first and second derivatives of f at P; explicitly, it is defined by the quadratic form given by the second jet of f at P after removing the linear term corresponding to the tangent line.

We refer to classical sources such as [Cay59, Cay65] for detailed derivations and to modern treatments in [MM19]. Higher order contact with conics gives rise to sextactic points.

Definition 1.8
Let P be a smooth point of C. The point P is called a *sextactic point* if there exists a conic Q such that

$$I_P(C, Q) \geq 6.$$

A conic realizing this maximal contact is called an *sextactic conic* at P.

Sextactic points have been studied since the nineteenth century (see [Cay59, Cay65]) and continue to play an important role in the geometry of plane curves; modern treatments can be found for instance in [MM19].

An important fact concerning sextactic points is the following.

Theorem 1.2 (Theorem C.1 [TU02])
Let C be a regular algebraic curve of degree d in $\mathbb{P}^2(\mathbb{C})$. Then C has exactly $3d(4d-9)$ sextactic points counted with multiplicities if all inflection points on C are simple. If C has k inflection points with multiplicities $\nu_1, \dots, \nu_k$, respectively, then C has

$$3d(5d-11) - \sum_{i=1}^{k}(4\nu_i - 3)$$

sextactic points counted with multiplicities.

More generally, one may consider higher order contact with curves of arbitrary degree. In particular, contact of order 9 with cubics appears naturally for elliptic curves and plays a role in the study of special points on Fermat curves; see [Gat79] for classical results.

The notions introduced in this section provide the language used in the sequel to describe sextactic points, points of type 9, and arrangements of osculating curves associated with the plane curves studied in this thesis.

1.4 Fermat curves

For an integer $n \geq 3$, the *Fermat curve* of degree n is the smooth plane curve

$$F_n\colon x^n + y^n + z^n = 0$$

in $\mathbb{P}^2(\mathbb{C})$.

It is well known that F_n is smooth. Indeed, if $f = x^n + y^n + z^n$, then the partial derivatives

$$f_x = nx^{n-1}, \qquad f_y = ny^{n-1}, \qquad f_z = nz^{n-1}$$

cannot vanish simultaneously at a point of $\mathbb{P}^2(\mathbb{C})$, hence $\mathrm{Sing}(F_n) = \varnothing$.

The Fermat curves possess a large group of projective automorphisms generated by permutations of the coordinates and by diagonal transformations

$$[x : y : z] \longmapsto [\lambda x : \mu y : z],$$

where $\lambda^n = \mu^n = 1$. This symmetry plays an important role in the study of special points and configurations associated with F_n.

In the sequel we will use the notation F_n throughout, with particular emphasis on the cases $n = 3$ and $n = 4$.

Special points on F_n, such as sextactic points and points of higher order contact with conics or cubics, will be studied in detail in subsequent chapters. Background on Fermat curves and their geometric properties can be found for instance in [Dol12, For81].

1.5 Arrangements of plane curves

In this thesis we consider configurations consisting of finitely many plane curves.

Definition 1.9
An *arrangement of plane curves* is a reduced divisor

$$\mathcal{A} = C_1 \cup \cdots \cup C_r \subset \mathbb{P}^2(\mathbb{C}),$$

where each $C_i = V(f_i)$ is an irreducible plane curve.

Equivalently, if $f = f_1 \cdots f_r \in S = \mathbb{C}[x, y, z]$, then the arrangement $\mathcal{A}$ is defined by the reduced equation

$$\mathcal{A} = V(f).$$

Definition 1.10
The *singular locus* of an arrangement $\mathcal{A}$ is the set

$$\mathrm{Sing}(\mathcal{A}) = \bigcup_{i<j} (C_i \cap C_j) \ \cup \ \bigcup_i \mathrm{Sing}(C_i).$$

Points of $\mathrm{Sing}(\mathcal{A})$ are called *singular points* of the arrangement.

In the situations studied in this thesis, the components C_i will typically be smooth curves, so singularities of the arrangement arise from intersections of distinct components.

Two arrangements $\mathcal{A}$ and $\mathcal{B}$ are said to be *projectively equivalent* if there exists an element of $\mathrm{PGL}(3, \mathbb{C})$ sending $\mathcal{A}$ onto $\mathcal{B}$. Classification results obtained later will be understood up to this equivalence.

Although arrangements may be studied from combinatorial or topological viewpoints, our approach is primarily algebraic and geometric. In particular, we will focus on the defining equation of the arrangement and on the singularities it produces. Background on arrangements of plane curves and related algebraic structures can be found for instance in [Sar10, DS14, Dim16].

1.6 Logarithmic derivations and freeness

Let $S = \mathbb{C}[x, y, z]$ and let $f \in S$ be a reduced homogeneous polynomial defining a plane curve or, more generally, an arrangement $\mathcal{A} = V(f) \subset \mathbb{P}^2(\mathbb{C})$.

Definition 1.11
The *Jacobian ideal* of f is the homogeneous ideal

$$J_f = (f_x, f_y, f_z) \subset S$$

generated by the partial derivatives of f.

The algebraic structure governing the singularities of $\mathcal{A}$ is encoded in the module of logarithmic derivations.

Definition 1.12
The *module of logarithmic derivations* along $\mathcal{A} = V(f)$ is

$$D(f) = \{\theta \in \mathrm{Der}_{\mathbb{C}}(S) \mid \theta(f) \in (f)\}.$$

Equivalently, $D(f)$ consists of polynomial vector fields on $\mathbb{P}^2(\mathbb{C})$ tangent to the curve $\mathcal{A}$. This module is graded and reflects both algebraic and geometric properties of the arrangement.

Definition 1.13
The curve (or arrangement) $\mathcal{A} = V(f)$ is called *free* if the S–module $D(f)$ is free of rank 3.

If $\mathcal{A}$ is free, then there exist homogeneous derivations forming a basis of $D(f)$, and the corresponding degrees are called the *exponents* of $\mathcal{A}$.

Freeness of plane curves and arrangements is closely related to the syzygies among the partial derivatives of f and to numerical invariants of the singularities. We refer to [DS14, Dim16, Dim87] for background and further developments.

Chapter 2

Type 9 points on the Fermat cubic

In this Chapter, we study the Fermat curves $F_n : x^n + y^n + z^n = 0$. We start with determining the coordinates of the sextactic points of F_n and derive equations of the osculating conics tangent to F_n at these points with multiplicity 6. Then we compute the exact coordinates of the points of type 9 on the Fermat conic F_3 and the equations of cubics tangent to F_3 at these points with multiplicity 9. At the end we investigate certain configurations of conics tangent to F_3 either at two sextactic points or at two points of type 9, with multiplicity 3. We also present the algorithm that can be used to find such bitangent conics.

Some results contained in this chapter have been already published in [MZ25a].

2.1 Sextactic points on Fermat curves

In this section, we compute the exact coordinates of sextactic points on a Fermat curve F_n of degree n ($n > 3$), given by the equation

$$F_n : x^n + y^n + z^n = 0.$$

First, we are going to determine the number of the sextactic points of F_n. Note that the first Hessian of F_n is given by

$$H(F_n) = x^{n-2}y^{n-2}z^{n-2}$$

up to a scalar. Therefore all inflection points of F_n lie on the coordinate lines $x = 0$, $y = 0$ and $z = 0$. Their exact coordinates can be written as

$$\left[0 : 1 : \varepsilon_{2n}^k\right], \quad \left[\varepsilon_{2n}^k : 0 : 1\right], \quad \left[1 : \varepsilon_{2n}^k : 0\right],$$

where $k \in \{0, 1, \ldots, n-1\}$ and $\varepsilon_{2n} \in \mathbb{C}$ is the primitive root of unity of order $2n$. We therefore have $3n$ inflection points on F_n. The respective lines tangent to F_n at these points are given by

$$y + \varepsilon_{2n}^{k(n-1)} z = 0, \quad \varepsilon_{2n}^{k(n-1)} x + z = 0, \quad x + \varepsilon_{2n}^{k(n-1)} y = 0.$$

It can be checked that these lines intersect F_n in exactly one point, thus they are tangent with multiplicity n. Therefore all inflection points have multiplicity $n-2$. Hence by Theorem 1.2 we have

$$3n(5n-11) - 3n\Big(4(n-2) - 3\Big) = 3n^2$$

sextactic points on F_n.

In order to compute their exact coordinates, we are going to use the method involving *the second Hessian* of a curve. The notion of the second Hessian was introduced by Cayley in 1865. For a given curve C, its second Hessian, denoted by $H_2(C)$, is a curve which contains all the sextactic points on C as well as all the inflection points.

In [Cay59], Cayley gives an explicit formula for second Hessian of a given curve. However, the formula contained a mistake, which was later corrected by Maugesten and Moe in [MM19].

In [SS24], this formula was presented in a particularly transparent way. It was then used to compute the second Hessian of a Fermat cubic F_3. We used the same method to compute the second Hessian of F_n for any $n \geqslant 3$. The obtained result was then double-checked in Singular for some small values of n (see Appendix, section 5.1). The obtained formula is presented below.

Proposition 2.1
The second Hessian of the Fermat curve $F_n : x^n + y^n + z^n = 0$, up to a scalar, is given by the following formula:

$$H_2(F_n) = (xyz)^{3n-9}(x^n - y^n)(y^n - z^n)(z^n - x^n).$$

From this formula it is now possible to determine the exact coordinates of all sextactic points on F_n. First of all, since the first Hessian of F_n is equal to $H(F_n) = (xyz)^{n-2}$ up to a constant, we know that the inflection points on F_n are precisely the points on F_n which lie on coordinate lines $x = 0$, $y = 0$ and $z = 0$. Therefore all the sextactic points on F_n (which are all the other points of intersection of F_n and $H_2(F_n)$) lie on a curve given by

$$(x^n - y^n)(y^n - z^n)(z^n - x^n) = 0.$$

By considering each factor of the equation above, it is now easy to compute all the coordinates of sextactic points. We obtain exactly $3n^2$ points, as desired. The computed coordinates are presented below.

Proposition 2.2
The sextactic points on $F_n : x^n + y^n + z^n = 0$ are

$$\left[1 : \varepsilon_{2n}^{2k} : \varepsilon_{2n}^{2\ell+1} \sqrt[n]{2}\right], \quad \left[\varepsilon_{2n}^{2\ell+1} \sqrt[n]{2} : 1 : \varepsilon_{2n}^{2k}\right], \quad \left[\varepsilon_{2n}^{2k} : \varepsilon_{2n}^{2\ell+1} \sqrt[n]{2} : 1\right],$$

where $k, l \in \{0, 1, \ldots, n-1\}$ and $\varepsilon_{2n} \in \mathbb{C}$ is the primitive root of unity of order $2n$.

In [Cay59] Cayley gives the formula for the conic tangent to the curve at a given point, which intersects the curve with multiplicity at least 5. We can use this formula to compute the equation of such a conic for a given point on the Fermat curve of degree n.

Proposition 2.3
Let $P = [x_0 : y_0 : z_0]$ be any point on F_n. The conic tangent to F_n at P with intersection multiplicity at least 5 is given by the following formula:

$$\begin{aligned} &9(n-1)^2 \left(x^2 x_0^{n-2} + y^2 y_0^{n-2} + z^2 z_0^{n-2}\right)(x_0 y_0 z_0)^n \\ &- 6(n-1)(n-2)\left(x y_0 z_0 + x_0 y z_0 + x_0 y_0 z\right)\left(x x_0^{n-1} + y y_0^{n-1} + z z_0^{n-1}\right)(x_0 y_0 z_0)^{n-1} \\ &- (n+1)(n-2)(x_0^n y_0^n + y_0^n z_0^n + z_0^n x_0^n)\left(x x_0^{n-1} + y y_0^{n-1} + z z_0^{n-1}\right)^2 = 0. \end{aligned}$$

Note that this equation holds for the inflection points on F_n as well. Indeed, if P is such a point, then $x_0 y_0 z_0 = 0$ and the above equation reduces to

$$\left(x x_0^{n-1} + y y_0^{n-1} + z z_0^{n-1}\right)^2 = 0,$$

and $x x_0^{n-1} + y y_0^{n-1} + z z_0^{n-1} = 0$ is the equation of a line tangent to F_n at P.

For better clarity, the above equation can be rewritten so that the coefficients at all monomials are clearly visible. Using the fact that $x_0^n + y_0^n + z_0^n = 0$, we obtain the new equation in the following form:

$$\begin{aligned} &(n+1)x_0^{2n-2}\left((2n-1)y_0^n z_0^n + (n-2)x_0^{2n}\right)x^2 \\ &+ (n+1)y_0^{2n-2}\left((2n-1)z_0^n x_0^n + (n-2)y_0^{2n}\right)y^2 \\ &+ (n+1)z_0^{2n-2}\left((2n-1)x_0^n y_0^n + (n-2)z_0^{2n}\right)z^2 \\ &- 2(n-2)x_0^{n-1}y_0^{n-1}\left((n+1)x_0^n y_0^n - (4n-2)z_0^{2n}\right)xy \\ &- 2(n-2)z_0^{n-1}x_0^{n-1}\left((n+1)z_0^n x_0^n - (4n-2)y_0^{2n}\right)zx \\ &- 2(n-2)y_0^{n-1}z_0^{n-1}\left((n+1)y_0^n r^n - (4n-2)x_0^{2n}\right)yz = 0 \end{aligned}$$

The configurations of these conics for Fermat curve F_n of any degree n were recently studied in [TM93].

2.2 Coordinates with division polynomials

In this section, we are going to compute the exact coordinates of all sextactic points and all type 9 points on the Fermat cubic F_3. The exact coordinates of sextactic points on F_3 were already computed in [SS24], but we are going to use a different method, which involves division polynomials.

In order to do so, we need some facts derived from Abel's Theorem. Below we state the theorem in a version sufficient for our purposes:

Theorem 2.4 (Abel's Theorem)
Let E be a smooth complex elliptic curve with distinguished point 0 embedded in the complex projective plane. Then a divisor $D = \sum d_i P_i$ on E is a (scheme theoretic) intersection of E with another plane curve C of degree c if and only if $\sum d_i = 3c$ and $\sum d_i P_i = 0$ in the group law on E.

As a result, we obtain the following corollary:

Corollary 2.1

1. Sextactic points on a cubic C are precisely the 6-torsion points on C, which are not the 3-torsion points.
2. Type 9 points on a cubic C are precisely the 9-torsion points on C, which are not the 3-torsion points.

Therefore, in order to find the coordinates of the sextactic points and the points of type 9 on the Fermat cubic F_3, we need to compute 6-torsion points and 9-torsion points of F_3 and then exclude the 3-torsion points from the obtained list.

It is well known that the set of 3-torsion points on Fermat cubic F_3 coincides with its flex points. Since the Hessian of F_3, up to a scalar, is

$$H(F_3) = xyz,$$

we know that flex points lie on coordinate lines $x = 0$, $y = 0$ and $z = 0$. Their exact coordinates are listed below (here $\varepsilon_3 \in \mathbb{C}$ denotes the primitive root of unity of order 3):

$$[1 : -1 : 0],\ [1 : 0 : -1],\ [0 : 1 : -1],$$
$$[1 : -\varepsilon_3 : 0],\ [1 : 0 : -\varepsilon_3],\ [0 : 1 : -\varepsilon_3],$$
$$[1 : -\varepsilon_3^2 : 0],\ [1 : 0 : -\varepsilon_3^2],\ [0 : 1 : -\varepsilon_3^2].$$

To compute the coordinates of the 6-torsion and 9-torsion points, we use the division polynomials. We are going to use the definition and properties based on [Eng99]. All necessary computations have been done using Singular (see Appendix, section 5.3).

For a given elliptic curve with equation written in the Weierstrass form $y^2 = x^3 + Ax + B$, the sequence of division polynomials $(\psi_n)_{n\in\mathbb{N}}$ is defined recursively in the following way:

$$\begin{aligned}
\psi_0 &= 0,\\
\psi_1 &= 1,\\
\psi_2 &= 2y,\\
\psi_3 &= 3x^4 + 6Ax^2 + 12Bx - A^2,\\
\psi_4 &= 4y(x^6 + 5Ax^4 + 20Bx^3 - 5A^2x^2 - 4ABx - 8B^2 - A^3),\\
\psi_{2k+1} &= \psi_{k+2}\psi_k^3 - \psi_{k-1}\psi_{k+1}^3 \text{ for } k \geq 2,\\
\psi_{2k} &= \frac{\psi_k}{2y}\left(\psi_{k+2}\psi_{k-1}^2 - \psi_{k-2}\psi_{k+1}^2\right) \text{ for } k > 2.
\end{aligned} \tag{2.1}$$

We are going to use the following fact.

Fact 2.1

Let E be an elliptic curve given by the equation $y^2 = x^3 + Ax + B$.

1. *The roots of ψ_{2k+1} (after replacing y^2 with $x^3 + Ax + B$) are the x-coordinates of the $(2k+1)$-torsion points of E, excluding the origin point.*

2. *The roots of $\frac{\psi_{2k}}{y}$ (after replacing y^2 with $x^3 + Ax + B$) are the x-coordinates of the $(2k)$-torsion points of E, excluding the 2-torsion points.*

In order to use this method for the Fermat cubic, we need to write its equation in the Weierstrass form first. This can be done under the following substitutions:

$$\tilde{x} = \frac{-12z}{x+y}, \quad \tilde{y} = \frac{36(x-y)}{x+y}, \tag{2.2}$$

As a result, we obtain the following equation:

$$\begin{aligned}
\tilde{y}^2 &= \frac{1296(x-y)^2}{(x+y)^2} = \frac{1296(x^3 + y^3 - x^2y - xy^2)}{(x+y)^3}\\
&= \frac{1728(x^3+y^3) - 432(x+y)^3}{(x+y)^3} = \frac{-1728z^3}{(x+y)^3} - 432 = \tilde{x}^3 - 432.
\end{aligned}$$

We therefore have $A = 0$ and $B = -432$.

We start with sextactic points on F_3. First, we use equations (2.1) to determine $\frac{\psi_6}{y}$ and substitute $y^2 = x^3 - 432$ (note that such a substitution is possible, because in the obtained polynomial $\frac{\psi_6}{y}$ all powers of y are even). After simplifying, we obtain the following polynomial of degree 16:

$$\begin{aligned}\varphi_6(x) = 6x^{16} - 580608x^{13} - 1003290624x^{10} + 3591207124992x^7 \\ - 374476218826752x^4 + 184884258895036416x.\end{aligned}$$

As previously stated in [SS24], the coordinates of sextactic points can be expressed over $\mathbb{Q}(\varepsilon_3, \mu)$, where $\mu = \sqrt[3]{2}$ and $\varepsilon_3 \in \mathbb{C}$ is again the primitive root of unity of order 3. It turns out that the obtained polynomial $\varphi_6(x)$ splits into linear factors over the same field. We can use Singular to split $\varphi_6(x)$ over $\mathbb{Q}(\varepsilon_3, \mu)$. However, in order to use this field in Singular, we need to find the primitive element of the field extension and its minimal polynomial. It turns out that $\mathbb{Q}(\varepsilon_3, \mu) = \mathbb{Q}(\varepsilon_3 + \mu)$ and the minimal polynomial of $\varepsilon_3 + \mu$ is of the form

$$\Gamma_6(x) = x^6 + 3x^5 + 6x^4 + 3x^3 + 9x + 9.$$

We obtain 16 distinct linear factors of $\varphi_6(x)$ and therefore we have 16 different roots (which are by Fact 2.1 the x-coordinates of the 6-torsion points on F_3). Since for each x we have exactly two different values of y (because $y^2 = x^3 - 432$ and for each x-coordinate we have $x^3 - 432 \neq 0$), we obtain 32 different points (x, y). However, by Fact 2.1 we also need to add 2-torsion points to the list. Apart from the origin, we have three nontrivial 2-torsion points on F_3 of the form $(u, 0)$, with $u^3 - 432 = 0$.

The next step is to go back to the projective case and therefore we need to apply the inverse of (2.2). The inverse is given by

$$x = 36 + \tilde{y}, \quad y = 36 - \tilde{y}, \quad z = -6\tilde{x}. \tag{2.3}$$

This way we obtain a list of coordinates of 35 points. After excluding all 3-torsion points from the list, we are left with the following coordinates:

$$
\begin{aligned}
S_1 &= [1 : -\tfrac{1}{2}\mu^2\varepsilon_3 : -\tfrac{1}{2}\mu^2\varepsilon_3], & S_{15} &= [1 : -\tfrac{1}{2}\mu^2\varepsilon_3 : -\tfrac{1}{2}\mu^2],\\
S_2 &= [1 : \mu(\varepsilon_3+1) : \mu], & S_{16} &= [1 : \mu(\varepsilon_3+1) : -\varepsilon_3 - 1],\\
S_3 &= [1 : -\mu : -\varepsilon_3 - 1], & S_{17} &= [1 : \varepsilon_3 : -\mu],\\
S_4 &= [1 : -\tfrac{1}{2}\mu^2 : \tfrac{1}{2}\mu^2(\varepsilon_3+1)], & S_{18} &= [1 : -\varepsilon_3 - 1 : \mu(\varepsilon_3+1)],\\
S_5 &= [1 : \tfrac{1}{2}\mu^2(\varepsilon_3+1) : -\tfrac{1}{2}\mu^2\varepsilon_3], & S_{19} &= [1 : \varepsilon_3 : \mu(\varepsilon_3+1)],\\
S_6 &= [1 : -\mu\varepsilon_3 : -\varepsilon_3 - 1], & S_{20} &= [1 : -\varepsilon_3 - 1 : -\mu\varepsilon_3],\\
S_7 &= [1 : \varepsilon_3 : -\mu\varepsilon_3], & S_{21} &= [1 : -\mu : 1],\\
S_8 &= [1 : -\varepsilon_3 - 1 : -\mu], & S_{22} &= [1 : -\tfrac{1}{2}\mu^2 : -\tfrac{1}{2}\mu^2],\\
S_9 &= [1 : -\mu : \varepsilon_3], & S_{23} &= [1 : \tfrac{1}{2}\mu^2(\varepsilon_3+1) : -\tfrac{1}{2}\mu^2],\\
S_{10} &= [1 : -\tfrac{1}{2}\mu^2 : -\tfrac{1}{2}\mu^2\varepsilon_3], & S_{24} &= [1 : -\mu\varepsilon_3 : \varepsilon_3],\\
S_{11} &= [1 + \tfrac{1}{2}\mu^2(\varepsilon_3+1) : \tfrac{1}{2}\mu^2(\varepsilon_3+1)], & S_{25} &= [1 : 1 : -\mu],\\
S_{12} &= [1 : -\mu\varepsilon_3 : 1], & S_{26} &= [1 : 1 : -\mu\varepsilon_3],\\
S_{13} &= [1 : -\tfrac{1}{2}\mu^2\varepsilon_3 : \tfrac{1}{2}\mu^2(\varepsilon_3+1)], & S_{27} &= [1 : 1 : \mu(\varepsilon_3+1)].\\
S_{14} &= [1 : \mu(\varepsilon_3+1) : \varepsilon_3], & &
\end{aligned}
$$

The following list of coordinates is identical to the list of coordinates of sextactic points presented in [SS24]. After some computations, it can also be verified that these coordinates coincide with the coordinates from Proposition 2.2 for $n = 3$ (using the equality $\varepsilon_6 = -\varepsilon_3^2$).

We can use similar method to compute the coordinates of the 9-torsion points on F_3. We use equations (2.1) to determine the polynomial ψ_9 and we again substitute $y^2 = x^3 - 432$. In this case we obtain the polynomial of degree 40 with the following leading terms:

$$\varphi_9(x) = 9x^{40} - 9859968x^{37} - 242429054976x^{34} + 7321604484759552x^{31} - \dots$$

The next step is to find a field over which $\varphi_9(x)$ splits into linear factors. It turns out that in this case we can use the field $\mathbb{Q}(\alpha, \beta) = \mathbb{Q}(\alpha + \beta)$, where $\alpha = \sqrt[3]{3}$ and $\beta = \varepsilon_9$ is the primitive root of unity of order 9.

As before, we need to find the minimal polynomial of $\gamma = \alpha + \beta$. We can use the method presented in the proof of [AS03, Theorem 2.9.1], which allows us to construct a polynomial with $\alpha + \beta$ as a root using the minimal polynomials of α and β, which are $A(x) = x^3 - 3$ and $B(x) = x^6 + x^3 + 1$, respectively. The obtained polynomial can be then factorized to obtain the minimal polynomial of $\alpha + \beta$. This computation can also be done in Singular (see Appendix, section 5.2).

As a result, we obtain the following polynomial:

$$\Gamma_9(x) = x^{18} - 15x^{15} + 177x^{12} - 578x^9 + 6747x^6 + 642x^3 + 343.$$

By factorizing the obtained polynomial $\varphi_9(x)$ over $\mathbb{Q}(\gamma)$, we get 40 x-coordinates. Since again for each x we have two different values of y, we obtain 80 different pairs (x, y). After applying the substitution (2.3), we get a list of 80 coordinates. Naturally, among them we still have 3-torsion points of F_3, which we need to remove. That way we finally obtain the coordinates of 72 points of type 9.

These coordinates can be written using rather complicated expressions involving γ, but they can be significantly simplified when we substitute $\gamma = \alpha + \beta$. All necessary computations have been done using Singular. Explicit coordinates of the first 12 points of type 9 on F_3 are presented below. The coordinates of the remaining 60 points can be obtained by permuting the coordinates of the first 12 points.

$$\begin{aligned}
T_1 &= [1 : \beta : \beta^2],\\
T_2 &= [1 : \beta : \beta^5],\\
T_3 &= [1 : \beta^2 : \beta^4],\\
T_4 &= [3 : \alpha\beta(\beta-1)(\beta^3-1) : -\alpha^2\beta(\beta^4+\beta^2+1)],\\
T_5 &= [3 : \alpha\beta(\beta-1)(\beta^3-1) : \alpha^2(\beta-1)(\beta^3+\beta^2+1)],\\
T_6 &= [3 : \alpha\beta(\beta-1)(\beta^3-1) : \alpha^2(\beta^5-\beta^4+\beta^3+\beta^2+1)],\\
T_7 &= [3 : \alpha\beta(\beta-1)(\beta^3+2) : -\alpha^2\beta(\beta^4+\beta^2+1)],\\
T_8 &= [3 : \alpha\beta(\beta-1)(\beta^3+2) : \alpha^2(\beta-1)(\beta^3+\beta^2+1)],\\
T_9 &= [3 : \alpha\beta(\beta-1)(\beta^3+2) : \alpha^2(\beta^5-\beta^4+\beta^3+\beta^2+1)],\\
T_{10} &= [3 : -\alpha\beta(\beta-1)(2\beta^3+1) : -\alpha^2\beta(\beta^4+\beta^2+1)],\\
T_{11} &= [3 : -\alpha\beta(\beta-1)(2\beta^3+1) : \alpha^2(\beta-1)(\beta^3+\beta^2+1)],\\
T_{12} &= [3 : -\alpha\beta(\beta-1)(2\beta^3+1) : \alpha^2(\beta^5-\beta^4+\beta^3+\beta^2+1)].
\end{aligned} \tag{2.4}$$

Since the set of flex points on F_3 and the set of sextactic points on F_3 are both complete intersections (with the Hessian $H(F_3) = xyz$ and the second Hessian $H_2(F_3) = (x^3 - y^3)(y^3 - z^3)(z^3 - x^3)$, respectively), it is natural to ask whether the set of points of type 9 on F_3 is also a complete intersection of F_3 and a curve of degree 24. In order to find the equation of such a curve, we take the ideal $I_T \subset \mathbb{Q}(\alpha, \beta)[x, y, z]$ of all the points of type 9 and compute its reduced standard basis in Singular. It turns out that this ideal indeed has two generators. We obtain the following result.

Proposition 2.5
The set of points of type 9 *on* F_3 *is a complete intersection of* $x^3 + y^3 + z^3 = 0$ *and a curve* $Q(x, y) = 0$*, where*

$$Q(x, y) = x^{24} + 4x^{21}y^3 - 17x^{18}y^6 - 65x^{15}y^9 - 89x^{12}y^{12} - 65x^9y^{15} - 17x^6y^{18} + 4x^3y^{21} + y^{24}.$$

It can be checked that the polynomial $Q(x, y)$ splits into 24 linear factors over $\mathbb{Q}(\alpha, \beta)$. Therefore all 72 points of type 9 on F_3 are distributed on 24 lines in $\mathbb{P}^2(\mathbb{C})$. It can be further verified that each line contains exactly 3 points of type 9 and since all 24 lines intersect in the single point $[0 : 0 : 1]$, each point of type 9 lies on exactly one line.

Additionally, it can be checked that we have

$$I_T = (F_3, Q(x, y)) = (F_3, Q(x, z)) = (F_3, Q(y, z)),$$

and therefore, by factoring the polynomials $Q(x, z)$ and $Q(y, z)$, we obtain two new sets of 24 lines, on which all points of type 9 are again equally distributed.

2.3 Special bitangent conics

First, we describe the algorithm which we use to find such conics. We start with the following lemma about intersection of two affine curves. The proof of this lemma is presented in [MZ25a].

Lemma 2.1 ([MZ25a], Lemma 4.2)
Let $f, g \in \mathbb{K}[x, y]$ *be two affine curves such that* $f(0, 0) = g(0, 0) = 0$*. If* f *and* g *can be written in the form*

$$f = ax + \textit{(remaining terms)},$$
$$g = by^k + \textit{(higher degree terms)}$$

with $a, b \in \mathbb{K}$*,* $a \neq 0$*,* $b \neq 0$ *and* $k \in \mathbb{N}$*, then* $I_{(0,0)}(f, g) \leq k$*.*

Let C be a smooth curve of degree at least 3 and let $P = [x_0 : y_0 : z_0]$ be a point on C. Let Q be a conic given by the general equation

$$Q : ax^2 + by^2 + cz^2 + dxy + exz + fyz = 0$$

for some $a, b, \ldots, f \in \mathbb{C}$. Assume that Q is tangent to C at P with multiplicity k, where $3 \leq k \leq \deg(C)$. We want to determine some conditions for the coefficients $a, b, \ldots, f$ of

Q. First, in order to use Lemma 2.1, we need to switch to affine case. Without loss of generality we may assume $z_0 = 1$. Then we can use substitutions

$$x \leftarrow x + x_0, \quad y \leftarrow y + y_0, \quad z \leftarrow 1$$

in C and Q to shift both equations to the point $(0, 0)$. We denote these new equations by C_0 and Q_0, respectively. In particular, we have

$$Q_0(0, 0) = 0. \tag{2.5}$$

Denote by $\mathrm{coef}_X(m)$ the coefficient at monomial m in X. Note that since C is a smooth curve, C_0 is smooth at $(0, 0)$ and therefore we have either $\mathrm{coef}_{C_0}(x) \neq 0$ or $\mathrm{coef}_{C_0}(y) \neq 0$. Without loss of generality we may assume that $\mathrm{coef}_{C_0}(x) \neq 0$. Let

$$Q_1 = Q_0 - \frac{\mathrm{coef}_{Q_0}(x)}{\mathrm{coef}_{C_0}(x)} C_0.$$

Note that $\mathrm{coef}_{Q_1}(x) = 0$. Moreover, we also have

$$\mathrm{coef}_{Q_1}(y) = 0,$$

because otherwise by Lemma 2.1 we would have $I_{(0,0)}(C_0, Q_0) = I_{(0,0)}(C_0, Q_1) \leq 1$. Therefore Q_1 does not have any monomials of degree smaller than 2. Next, since $\mathrm{coef}_{C_0}(x) \neq 0$, there exists a polynomial $R_1 \in \mathbb{C}[x, y]$ of degree 1 such that the polynomial

$$Q_2 = Q_1 + R_1 C_0$$

does not contain monomials xy and x^2. Then we also have

$$\mathrm{coef}_{Q_2}(y^2) = 0,$$

because otherwise by Lemma 2.1 we would have $I_{(0,0)}(C_0, Q_0) = I_{(0,0)}(C_0, Q_2) \leq 2$.

For $k > 3$ this process can be repeated further, constructing inductively a sequence of polynomials $Q_3, \ldots, Q_{k-1}$. For $i \in \{3, \ldots, k-1\}$, in order to construct the polynomial Q_i, we choose $R_{i-1} \in \mathbb{C}[x, y]$ such that

$$Q_i = Q_{i-1} + R_{i-1} C_0$$

does not contain any monomial of degree i divisible by x. Then we also have

$$\mathrm{coef}_{Q_i}(y^i) = 0,$$

because otherwise by Lemma 2.1 we would have $I_{(0,0)}(C_0, Q_0) = I_{(0,0)}(C_0, Q_i) \leq i < k$.

At the end, we obtain k conditions necessary for the conic Q to exist, namely $Q_0(0,0) = 0$ and $\mathrm{coef}_{Q_i}(y^i) = 0$ for $i \in \{1, 2, \ldots, k-1\}$. All these conditions can be expressed as polynomial equations of variables a, b, c, d, e and f. Naturally, these conditions only refer to tangency at one specific point on C. In order to find a conic tangent to C at two points P_1 and P_2 with multiplicity at least k in each point, we need to write separate conditions for each point, obtaining a set of $2k$ polynomial equations.

The above algorithm can be naturally used for curves of any degree greater or equal to 3. However, for curves of degree 3, it is possible to obtain a simple criterion for such a conic to exist, which is a consequence of Abel's theorem (Theorem 2.4). This criterion is stated below (see [MZ25a, Proposition 4.1] for details):

Proposition 2.6
Let P_1 and P_2 be distinct points on a smooth elliptic curve E. Then there exists a conic Q tangent to E at these two points with multiplicity 3 in each point if and only if $P_1 + P_2$ is a 3-torsion point on E.

Example 2.7
Consider two sextactic points $P_1 = \left[-\sqrt[3]{2} : 1 : 1\right]$ and $P_2 = \left[1 : 1 : -\sqrt[3]{2}\right]$ on F_3. The line passing through both points is given by equation $x + z + \left(\sqrt[3]{2} - 1\right) y = 0$ and it also passes through the point $P_3 = [1 : 0 : -1]$, which is a flex point on F_3. Therefore there exists a conic tangent to F_3 at P_1 and P_2 with multiplicity 3 at both points. Indeed, using Singular and the algorithm described above, we can find the equation of this conic:

$$\sqrt[3]{2}x^2 + \sqrt[3]{2}z^2 + (\sqrt[3]{2} - 1)y^2 + xy + yz + \left(\sqrt[3]{4} + 1\right) xz = 0.$$

Remark 2.1
The above proposition indicates that for a generic point P on a curve C of degree 3 the number of bitangent conics passing through P is equal to the number of flex points on C. In particular, for a generic point P on the Fermat cubic F_3, we expect to find 9 conics bitangent to F_3 passing through it.

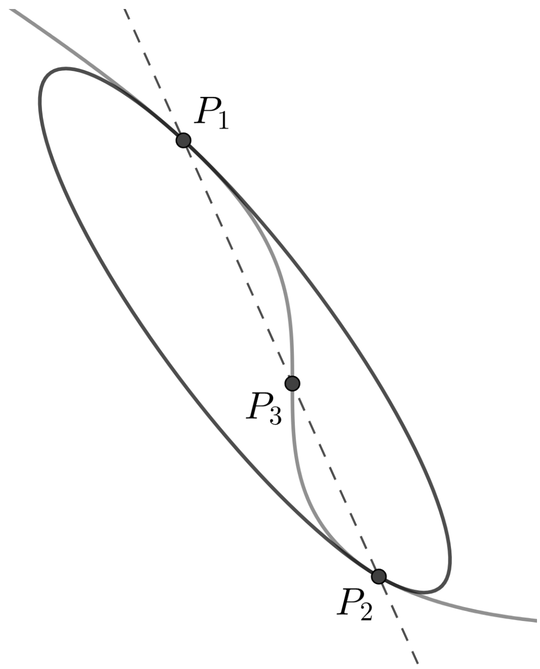


Figure 2.1: A bitangent conic from Example 2.7

2.4 Configurations of conics bitangent to F_3

In this section, we find all conics bitangent to F_3, passing through either two sextactic points or two points of type 9. We use the algorithm described in the previous section. All computations were done using Singular (see Appendix, section 5.4). We obtained the following result:

Proposition 2.8

a) There are exactly 108 *conics passing through two different sextactic points on* F_3*, intersecting* F_3 *at these points with multiplicities* 3*. Moreover, for each sextactic point, there are exactly* 8 *such conics passing through it.*

b) There are exactly 324 *conics passing through two different points of type* 9 *on* F_3*, intersecting* F_3 *at these points with multiplicities* 3*. Moreover, for each point of type* 9*, there are exactly* 9 *such conics passing through it.*

It is worth noting that for each point of type 9 on F_3 we obtain the expected number of 9 bitangent conics passing through it (see Remark 2.1), while we only obtain 8 conics passing through each sextactic point. This can be explained by the following observation: for each sextactic point S on F_3 there exists a flex point P on F_3 such that the line passing through S and P is tangent to F_3 at S. Moreover, it can be shown that for each flex point P on F_3, there exist exactly three sextactic points on F_3 such that the lines tangent to F_3 at these points pass through P.

In the following part, we are going to study the intersection patterns of the obtained bitangent conics. We are going to focus on conics passing through a specific point P on F_3, which is either a sextactic point or a point of type 9.

Note that since all obtained conics are tangent to F_3 at P with multiplicity 3, each two such conics are also tangent at P with multiplicity 3 and therefore they intersect in only one additional point besides P. We can therefore expect to have $\binom{8}{2} = 28$ additional points of intersection when P is a sextactic point and $\binom{9}{2} = 36$ points when P is a point of type 9. However, the actual number of intersection points can be smaller, because one such a point can be a point of intersection of more than two conics.

We start with P being a sextactic point on F_3. We obtain the following results:

Proposition 2.9
The bitangent conics passing through a given sextactic point S on Fermat cubic F_3 intersect in 24 different points besides S. In particular:

- *there are 22 points at which 2 such conics intersect,*
- *there are 2 points at which 3 such conics intersect,*
- *all 24 points of intersection do not lie on F_3.*

We can now gather all these obtained points of intersection for all 27 sextactic points on F_3. Then, we verify how many bitangent conics pass through each point.

Proposition 2.10
There are exactly 540 points not lying on Fermat cubic F_3 which are the intersection points of at least two bitangent conics passing through the same sextactic point on F_3. In particular:

- *there are 486 points at which 2 bitangent conics intersect,*
- *there are 36 points $A_1, A_2, \ldots, A_{36}$ at which 6 bitangent conics intersect,*
- *there are 18 points $B_1, B_2, \ldots, B_{18}$ at which 9 bitangent conics intersect.*

Similarly to sextactic points, the coordinates of points $A_1, A_2, \ldots, A_{36}$ as well as $B_1, B_2, \ldots, B_{18}$ can be written using ε_3 and μ. Explicit coordinates of the points $A_1, \ldots, A_8$ are presented below. The coordinates of the remaining 28 points can be obtained by permuting the coordinates of the first 8 points.

$$\begin{aligned}
A_1 &= [1 : 1 : 0], & A_5 &= [1 : \varepsilon_3 : 0],\\
A_2 &= [1 : 1 : -\mu^2], & A_6 &= [1 : \varepsilon_3 : -\mu^2],\\
A_3 &= [1 : 1 : -\varepsilon_3\mu^2], & A_7 &= [1 : \varepsilon_3 : -\varepsilon_3\mu^2],\\
A_4 &= [1 : 1 : -\varepsilon_3^2\mu^2], & A_8 &= [1 : \varepsilon_3 : -\varepsilon_3^2\mu^2].
\end{aligned}$$

In a similar way, we can write the coordinates of the first three points from the set $B_1, B_2, \ldots, B_{18}$:

$$B_1 = [1 : -\mu : 0], \qquad B_2 = [1 : -\varepsilon_3\mu : 0], \qquad B_3 = [1 : -\varepsilon_3^2\mu : 0].$$

It is worth noting that the points from the set $B_1, \ldots, B_{18}$ lie on three coordinate lines $x = 0$, $y = 0$ and $z = 0$, with six points on each line. Moreover, it can be checked that this set of points is a complete intersection of $xyz = 0$ and the curve of degree 6, given by

$$x^6 + y^6 + z^6 + \frac{5}{2}(x^3y^3 + y^3z^3 + z^3x^3) = 0.$$

We can now repeat the same procedure, but this time, we consider bitangent conics passing through a point of type 9 on F_3.

Proposition 2.11

The bitangent conics passing through a given point T of type 9 on Fermat cubic F_3 intersect in 30 different points besides T. In particular:

- *there are 27 points at which 2 such conics intersect,*
- *there are 3 points at which 3 such conics intersect,*
- *all 30 points of intersection do not lie on F_3.*

By gathering all the obtained points of intersection for all 72 points of type 9 on F_3, we obtain the following result:

Proposition 2.12

There are exactly 2016 points not lying on Fermat cubic F_3 which are the intersection points of at least two bitangent conics passing through the same point of type 9 on F_3. In particular:

- *there are 1944 points at which 2 bitangent conics intersect,*
- *there are 72 points $C_1, C_2, \ldots, C_{72}$ at which 9 bitangent conics intersect.*

Similarly to the points of type 9, the coordinates of the points $C_1, C_2, \ldots, C_{72}$ can be written using α and β. The coordinates of the first 12 points are presented below. Again, by permuting the coordinates of these points, we can obtain the coordinates of the remaining 60 points.

$$C_1 = [1 : \beta : 0], \qquad C_7 = \Big[3 : \alpha\beta(\beta-1)^2(\beta^2+\beta+1) : 0\Big],$$

$$C_2 = [1 : \beta^2 : 0], \qquad C_8 = \Big[3 : \alpha^2(\beta-1)^2(\beta+1) : 0\Big],$$

$$C_3 = [1 : \beta^4 : 0], \qquad C_9 = \Big[3 : -\alpha\beta(2\beta^4+\beta^3+\beta+2) : 0\Big],$$

$$C_4 = \Big[3 : \alpha\beta(\beta-1)(\beta^3+2) : 0\Big], \qquad C_{10} = \Big[3 : -\alpha^2\beta^2(\beta^2+\beta+1) : 0\Big],$$
$$C_5 = \Big[3 : -\alpha\beta(2\beta^4+\beta^3+\beta-1) : 0\Big], \qquad C_{11} = \Big[3 : \alpha\beta(\beta^4-\beta^3-\beta-2) : 0\Big],$$
$$C_6 = \Big[3 : \alpha\beta(\beta^4+2\beta^3-\beta+1) : 0\Big], \qquad C_{12} = \Big[3 : \alpha\beta(\beta^4-\beta^3+2\beta+1) : 0\Big].$$

The points $C_1, C_2, \dots, C_{72}$ lie on the coordinate lines $x = 0$, $y = 0$ and $z = 0$, with 24 points on each line. Moreover it can be further verified, that these points lie on the curve

$$Q(x, y) \cdot Q(x, z) \cdot Q(y, z) = 0,$$

where $Q(x, y)$ is the polynomial described in Proposition 2.5. More precisely, the points from the set $\{C_1, \dots, C_{72}\}$ which lie on the line $x = 0$ are precisely the intersection points of $x = 0$ and $Q(y, z) = 0$. Analogously, the points lying on $y = 0$ (respectively $z = 0$) are the intersection points of $y = 0$ and $Q(x, z) = 0$ (respectively $z = 0$ and $Q(x, y) = 0$).

2.5 Cubics tangent to points of type 9

By definition, if P is a point of type 9 on F_3, there exists an irreducible cubic tangent to F_3 at P with multiplicity 9. We have already computed the coordinates of all points of type 9 on F_3. The next step is to find a way to determine the equations of the tangent cubics.

If S is a sextactic point on F_3, the equation of an irreducible conic tangent to F_3 at S with multiplicity 6 can be determined using the formula shown in Proposition 2.3. For $S = [x_0 : y_0 : z_0]$, we obtain the equation

$$9\left(x^2x_0 + y^2y_0 + z^2z_0\right)(x_0y_0z_0)^3 - 3\,(xy_0z_0 + x_0yz_0 + x_0y_0z)\left(xx_0^2 + yy_0^2 + zz_0^2\right)(x_0y_0z_0)^2$$
$$- \,(x_0^3y_0^3 + y_0^3z_0^3 + z_0^3x_0^3)\left(xx_0^2 + yy_0^2 + zz_0^2\right)^2 = 0.$$

Note that for each sextactic point S of F_3 there is a unique irreducible conic tangent to F_3 at S with multiplicity 6.

For the points of type 9 on F_3, we can use the modified version of the algorithm presented in Section 2.3, used to find the equations of the bitangent conics. In this case, instead of searching for curves of degree 2 tangent to two given points with multiplicity at least 3 in both points, we search for curves of degree 3, tangent to one given point with multiplicity at least 9.

Below we present the results obtained for the first 12 points $T_1, \dots, T_{12}$ of type 9 (their exact coordinates are shown in (2.4) in Section 2.2). All these results were again obtained using Singular. Here, the cubic given by equation $G_i(x, y, z) = 0$ is tangent to the Fermat cubic at T_i, for $i \in \{1, 2, \ldots, 12\}$:

$$
\begin{aligned}
G_1(x,y,z) &= x^3 - \beta^3 y^3 - 3\beta^2 x^2 y + 3\beta x y^2 + 3\beta x^2 z + 3\beta^2(\beta^3+1)xz^2 \\
&\quad + 3\beta^2(\beta^3+1)y^2 z - 3\beta(\beta^3+1)yz^2 + (2\beta^3+1)xyz, \\
G_2(x,y,z) &= x^3 - \beta^3 y^3 - 3\beta^2 x^2 y + 3\beta x y^2 - 3\beta(\beta^3+1)x^2 z - 3\beta^2 xz^2 \\
&\quad - 3\beta^5 y^2 z + 3\beta yz^2 - (\beta^3-1)xyz, \\
G_3(x,y,z) &= x^3 + (\beta^3+1)y^3 - 3\beta^4 x^2 y + 3\beta^2 xy^2 + 3\beta^2 x^2 z + 3\beta(\beta^3+1)xz^2 \\
&\quad + 3\beta(\beta^3+1)y^2 z + 3\beta^5 yz^2 - (2\beta^3+1)xyz, \\
G_4(x,y,z) &= U_1(x,y,z) + V_1(x,y,z) + W_1(x,y,z) + 9\beta(\beta^4+\beta^3-\beta^2+\beta+1)xyz, \\
G_5(x,y,z) &= U_1(x,y,z) + V_2(x,y,z) + W_2(x,y,z) + 9(\beta-1)^2(\beta+1)xyz, \\
G_6(x,y,z) &= U_1(x,y,z) + V_3(x,y,z) + W_3(x,y,z) - 9(\beta^2+\beta+1)(\beta^3-\beta+1)xyz, \\
G_7(x,y,z) &= U_2(x,y,z) + V_1(x,y,z) + W_3(x,y,z) + 9(\beta-1)^2(\beta+1)xyz, \\
G_8(x,y,z) &= U_2(x,y,z) + V_2(x,y,z) + W_1(x,y,z) - 9(\beta^2+\beta+1)(\beta^3-\beta+1)xyz, \\
G_9(x,y,z) &= U_2(x,y,z) + V_3(x,y,z) + W_2(x,y,z) + 9\beta(\beta^4+\beta^3-\beta^2+\beta+1)xyz, \\
G_{10}(x,y,z) &= U_3(x,y,z) + V_1(x,y,z) + W_2(x,y,z) - 9(\beta^2+\beta+1)(\beta^3-\beta+1)xyz, \\
G_{11}(x,y,z) &= U_3(x,y,z) + V_2(x,y,z) + W_3(x,y,z) + 9\beta(\beta^4+\beta^3-\beta^2+\beta+1)xyz, \\
G_{12}(x,y,z) &= U_3(x,y,z) + V_3(x,y,z) + W_1(x,y,z) + 9(\beta-1)^2(\beta+1)xyz.
\end{aligned}
$$

The forms $U_i(x,y,z)$, $V_i(x,y,z)$ and $W_i(x,y,z)$ for $i \in \{1,2,3\}$ are defined as follows:

$$
\begin{aligned}
U_1(x,y,z) &= 9x^3 - 9\beta(\beta^3-\beta+1)y^3 - \alpha^2(\beta^5-\beta^4+\beta^3+\beta^2+1)x^2 y \\
&\quad + 3\alpha\beta(2\beta^4+\beta^3-3\beta^2+\beta+2)xy^2, \\
U_2(x,y,z) &= 9x^3 - 9\beta(\beta^3-\beta+1)y^3 - \alpha^2(\beta-1)(\beta^3+\beta^2+1)x^2 y \\
&\quad - 3\alpha(\beta^5+2\beta^4-\beta^2+\beta+3)xy^2, \\
U_3(x,y,z) &= 9x^3 - 9\beta(\beta^3-\beta+1)y^3 + \alpha^2\beta(\beta^2-\beta+1)(\beta^2+\beta+1)x^2 y \\
&\quad - 3\alpha(\beta^5-\beta^4-3\beta^3+2\beta^2+\beta-3)xy^2, \\
V_1(x,y,z) &= -3\alpha\beta(\beta^4-\beta^3+2\beta+1)x^2 z - \alpha^2\beta(2\beta^4-\beta^2+2)xz^2, \\
V_2(x,y,z) &= -3\alpha\beta(\beta^4+2\beta^3-\beta+1)x^2 z + \alpha^2(2\beta^4-2\beta^2+2\beta+1)xz^2, \\
V_3(x,y,z) &= 3\alpha\beta(2\beta^4+\beta^3+\beta+2)x^2 z + \alpha^2(\beta-1)(2\beta^4-\beta^2+\beta-1)xz^2, \\
W_1(x,y,z) &= -\alpha^2(\beta^4-\beta^2+\beta+2)y^2 z - 3\alpha\beta(\beta-1)(\beta^3+2)yz^2, \\
W_2(x,y,z) &= \alpha^2\beta(\beta-1)^2(\beta+1)^2 y^2 z + 3\alpha\beta(\beta-1)(2\beta^3+1)yz^2, \\
W_3(x,y,z) &= -\alpha^2(\beta^5-\beta^4-2\beta^3+\beta^2-2)y^2 z - 3\alpha\beta(\beta-1)^2(\beta^2+\beta+1)yz^2.
\end{aligned}
$$

Since the coordinates of the remaining points $T_{13}, T_{14}, \ldots, T_{72}$ of type 9 can be obtained by permuting the coordinates of the first 12 points, the equations of the cubics tangent to F_3 at these points with multiplicity 9 can be obtained from the above equations, by applying the respective permutations of variables.

It is worth noting that the cubic tangent to F_3 with multiplicity 9 at a given point of type 9 is not unique. In fact, based on the properties of the intersection index, for each $i \in \{1, 2, \ldots, 12\}$ and for each $\lambda \in \mathbb{C}$, the cubic given by the equation

$$G_i(x, y, z) + \lambda(x^3 + y^3 + z^3) = 0$$

is tangent to F_3 at T_i with multiplicity 9.

Chapter 3

Quartics with maximal number of maximal tangency points

In this chapter, we consider two quartics: the Fermat quartic F_4 and the Komiya-Kuribayashi quartic. These quartics have the maximal number of *maximal tangency lines* (and *maximal tangency points*). We study some line configurations related to these lines and points. We then compute the exact coordinates of the sextactic points on both quartics and we analyze the arrangements of conics tangent at two of these points with multiplicity 4. The results in this chapter are an extended version of the results from the article [MZ25b].

3.1 Definitions and notations

We start with the definition of a maximal tangency lines and points. They can be defined for an arbitrary plane curve.

Definition 3.1
Let $C \subset \mathbb{P}^2(\mathbb{C})$ be a smooth plane curve of degree d. We say that L is a *maximal tangency line* (MTL for short) of C, if it intersects C in exactly one point P. We say then that P is the *maximal tangency point* (MTP for short) of C.

In other words a point P is a maximal tangency point of a curve $C \subset \mathbb{P}^2(\mathbb{C})$ of degree d if and only if the line tangent to C at P is tangent to that curve with multiplicity d.

For plane curves of degree 4, the number of maximal tangency lines ranges from 0 to 12. It was proven in [KK79] that there are exactly two plane quartics with 12 maximal tangency lines, namely the Fermat quartic

$$F_4 : x^4 + y^4 + z^4 = 0$$

and the Komiya-Kuribayashi quartic

$$K : x^4 + x^4 + z^4 + 3(x^2y^2 + y^2z^2 + z^2x^2) = 0.$$

In the following sections, we are going to study the arrangements of MTLs and MTPs for these two quartics in more detail.

Below we state the definition of some specific operators, introduced by Roulleau in [Rou26] as a systematic way to construct new line arrangements and point configurations using the properties of the given ones.

Definition 3.2 (Roulleau)
For a subset $\mathbf{n} \subset \mathbb{N} \setminus \{0, 1\}$ and an arrangement of lines L in the plane, we define $\mathcal{P}_{\mathbf{n}}(L)$ as a set of points where exactly k lines from L intersect, for some $k \in \mathbf{n}$.

Definition 3.3 (Roulleau)
For a subset $\mathbf{n} \subset \mathbb{N} \setminus \{0, 1\}$ and a set of points P in the plane, we define $\mathcal{L}_{\mathbf{n}}(P)$ as a set of all lines containing exactly k points from P, for some $k \in \mathbf{n}$.

In order to simplify the notation, when $\mathbf{n} = \{m\}$ for some $m \in \mathbb{N}$, we will write $\mathcal{P}_m$ and $\mathcal{L}_m$ instead of $\mathcal{P}_{\{m\}}$ and $\mathcal{L}_{\{m\}}$.

Definition 3.4 (Roulleau)
We define $\mathcal{D}$ as an operator which maps a point $[a : b : c] \in \mathbb{P}^2(\mathbb{C})$ to a line in $\mathbb{P}^2(\mathbb{C})$ given by the equation $ax + by + cz = 0$. We also define the operator $\check{\mathcal{D}}$ as the inverse of $\mathcal{D}$.

Here we also recall the following definition of a combinatorial invariant of line arrangements. Let $\mathbf{L} = \{L_1, \ldots, L_d\}$ be an arrangement of d lines. Let r be the maximal multiplicity of $\mathbf{L}$, which is defined as

$$r = \max\{k \in \mathbb{N} : \text{ the arrangement } \mathbf{L} \text{ admits } k\text{-fold points}\}.$$

The t-vector of $\mathbf{L}$ is then defined as the vector

$$t(\mathbf{L}) = (t_2, t_3, \ldots, t_r),$$

where t_i is the number of points in $\mathbb{P}^2(\mathbb{C})$, where exactly i lines from $\mathbf{L}$ intersect.

3.2 The Fermat quartic

The maximal tangency points $PF = \{PF_1, \ldots, PF_{12}\}$ on the Fermat quartic F_4 are the inflection points, i.e. the points which lie on the first Hessian of F_4, which is equal to $H(F_4) = x^2y^2z^2$ (up to a constant). Therefore the points from PF form a complete intersection of curves $xyz = 0$ and $x^4 + y^4 + z^4 = 0$. By computing the exact coordinates of the points of intersection of F_4 and $H(F_4)$, we obtain

$$\begin{aligned} PF_1 &= [0:1:\varepsilon_8], & PF_5 &= [\varepsilon_8:0:1], & PF_9 &= [1:\varepsilon_8:0],\\ PF_2 &= [0:1:\varepsilon_8^3], & PF_6 &= [\varepsilon_8^3:0:1], & PF_{10} &= [1:\varepsilon_8^3:0],\\ PF_3 &= [0:1:\varepsilon_8^5], & PF_7 &= [\varepsilon_8^5:0:1], & PF_{11} &= [1:\varepsilon_8^5:0],\\ PF_4 &= [0:1:\varepsilon_8^7], & PF_8 &= [\varepsilon_8^7:0:1], & PF_{12} &= [1:\varepsilon_8^7:0], \end{aligned}$$

where $\varepsilon_8 \in \mathbb{C}$ is the primitive root of unity of order 8.

We start with the following interesting property.

Proposition 3.1

The 4 maximal tangency points of the Fermat quartic sitting on each of the coordinate axes form a harmonic range.

PROOF. By symmetry, it suffices to show it for one axis only. The maximal tangency points of the Fermat quartic which lie on the axis $z = 0$ have coordinates

$$[1:\varepsilon_8:0],\ [1:\varepsilon_8^3:0],\ [1:\varepsilon_8^5:0],\ [1:\varepsilon_8^7:0].$$

We can multiply the y-coordinate of these points by ε_8, which does not change their cross-ratio, because it corresponds to the automorphism of $\mathbb{P}^2(\mathbb{C})$ given by

$$[x:y:z] \longmapsto [x:y\cdot\varepsilon_8^7:z].$$

As a result, we obtain the coordinates

$$[1:i:0],\ [1:-1:0],\ [1:-i:0],\ [1:1:0].$$

In affine coordinates on the y-axis the above points correspond to 1, i, -1 and $-i$ and it is well known that they are a harmonic range. ■

There is yet another way to construct a symmetric arrangement of 12 lines based on the MTPs. The 12 maximal tangency points on the Fermat quartic can be divided into 4 groups of three points, where each group is of the form

$$\Big\{[0:1:\varepsilon_8^k],\ [\varepsilon_8^k:0:1],\ [1:\varepsilon_8^k:0]\Big\}$$

for $k \in \{1, 3, 5, 7\}$. In each such a group, the points lie on three lines, given by

$$\begin{aligned} x + \varepsilon_8^{2k} y - \varepsilon_8^k z &= 0, \\ y + \varepsilon_8^{2k} z - \varepsilon_8^k x &= 0, \\ z + \varepsilon_8^{2k} x - \varepsilon_8^k y &= 0. \end{aligned} \tag{3.1}$$

Gathering all such lines, we obtain a configuration of 12 lines. This configuration is shown on Figure 3.1, where each color represents lines for different values of k.

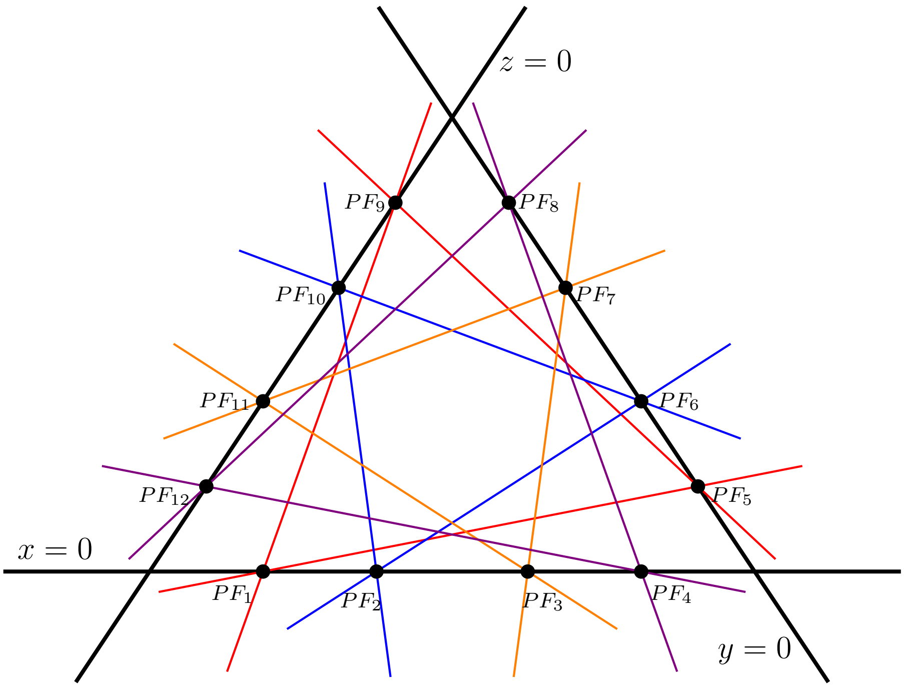


Figure 3.1: Configuration of 12 lines passing through the MTPs on the Fermat quartic $x^4 + y^4 + z^4$.

However, it turns out that even though this configuration is symmetric, these 12 lines intersect in 66 distinct points and hence there is no intersection point with multiplicity higher than 2.

Proposition 3.2

Let $S = \mathbb{C}[x, y, z]$. Let $\mathcal{I}_1$ be the saturated ideal of the 66 intersection points of lines given by (3.1). Then $\mathcal{I}_1$ is generated in degree 11 and its minimal free resolution has the following shape:

$$0 \longrightarrow S(-12)^{11} \xrightarrow{\alpha} S(-11)^{12} \longrightarrow S \longrightarrow S/\mathcal{I}_3 \longrightarrow 0.$$

The obtained resolution is linear and therefore the mapping α is given by the 12×11 matrix of linear forms. By the Hilbert-Burch Theorem [Eis95, Theorem 20.15], the generators of the ideal $\mathcal{I}_3$ can be chosen as determinants of the minors of order 11 of α.

We can study the intersection points of the above configuration in a bit more detail. It turns out that this set of 66 points can be divided into four groups:

- 12 maximal tangency points $PF_1, \ldots, PF_{12}$,
- a set of 6 points which are a complete intersection of curves $xyz = 0$ and the Fermat conic $F_2 : x^2 + y^2 + z^2 = 0$,
- a set of 24 points, contained in the Fermat arrangement of lines, determined by linear factors of $(x^2 - y^2)(y^2 - z^2)(z^2 - x^2)$,
- another set of 24 points, a complete intersection of curves given by

$$(xy + yz + 2xz)(yz + xz + 2xy)(xz + xy + 2yz) = 0,$$
$$x^4 + y^4 + z^4 + 4(x^2y^2 + y^2z^2 + z^2x^2) + 6xyz(x + y + z) = 0.$$

We can now focus on the maximal tangency lines for the Fermat quartic F_4. For each of the points $PF_1, \ldots, PF_{12}$ we can compute the equation of a line tangent to F_4 at this point, obtaining the list $LF = \{LF_1, \ldots, LF_{12}\}$ of maximal tangency lines. We get the following list of equations:

$$\begin{aligned} LF_1 &: y + \varepsilon_8^3 z = 0, & LF_5 &: \varepsilon_8^3 x + z = 0, & LF_9 &: x + \varepsilon_8^3 y = 0,\\ LF_2 &: y + \varepsilon_8 z = 0, & LF_6 &: \varepsilon_8 x + z = 0, & LF_{10} &: x + \varepsilon_8 y = 0,\\ LF_3 &: y + \varepsilon_8^7 z = 0, & LF_7 &: \varepsilon_8^7 x + z = 0, & LF_{11} &: x + \varepsilon_8^7 y = 0,\\ LF_4 &: y + \varepsilon_8^5 z = 0, & LF_8 &: \varepsilon_8^5 x + z = 0, & LF_{12} &: x + \varepsilon_8^5 y = 0. \end{aligned}$$

Here for each $k \in \{1, \ldots, 12\}$ the line LF_k is tangent to F_4 at PF_k. It can be checked that these lines are the linear factors of the polynomial

$$(x^4 + y^4)(y^4 + z^4)(z^4 + x^4).$$

It's also worth mentioning that in this case the maximal tangency points are dual to the maximal tangency lines, i.e. we have the equality $\mathcal{D}(PF) = LF$. More precisely, by looking at individual lines and points, we have $\mathcal{D}(PF_{2k}) = LF_{2k-1}$ and $\mathcal{D}(PF_{2k-1}) = LF_{2k}$ for $k \in \{1, \ldots, 6\}$.

We start with the properties of the line arrangement LF.

Proposition 3.3
The maximal tangency lines $LF_1, LF_2, \ldots, LF_{12}$ intersect in 48 double points and 3 points of multiplicity 4, i.e. we have the equality

$$t(LF) = (48, 0, 3).$$

It can be checked that the three points of multiplicity 4 are the coordinate points $[1:0:0]$, $[0:1:0]$ and $[0:0:1]$. The coordinates of the remaining 48 points, i.e. points from the set $\mathcal{P}_2(LF)$, can be determined by pairs of lines passing through distinct coordinate points. They can be written in the form

$$[1 : \varepsilon_8^k : \varepsilon_8^\ell],$$

where $k, \ell \in \{0, 1, \ldots, 7\}$ and at least one of the numbers k and ℓ is odd.

We now consider the ideal of points from $\mathcal{P}_2(LF)$ and study its minimal set of generators as well as its free resolution. We obtain the following proposition.

Proposition 3.4

Let $S = \mathbb{C}[x, y, z]$. Let $\mathcal{I}_2$ be the saturated ideal of the 48 points from $\mathcal{P}_2(LF)$. Then $\mathcal{I}_2$ is generated in degree 8 and its minimal free resolution has the following shape:

$$0 \longrightarrow S(-12)^2 \longrightarrow S(-8)^3 \longrightarrow S \longrightarrow S/\mathcal{I}_2 \longrightarrow 0.$$

Moreover, the three generators of the ideal $\mathcal{I}_2$ can be written in the following form:

$$(x^4 + y^4)(y^4 + z^4),\ (y^4 + z^4)(z^4 + x^4),\ (z^4 + x^4)(x^4 + y^4).$$

3.3 The Komiya-Kuribayashi quartic

The equations of the maximal tangency lines for the Komiya-Kuribayashi quartic were already computed explicitly by Edge in [Edg45]:

$$\begin{aligned}
&LK_1 : 2ix - y - z = 0, && LK_5 : 2iy - x - z = 0, && LK_9 : 2iz - x - y = 0,\\
&LK_2 : 2ix - y + z = 0, && LK_6 : 2iy - x + z = 0, && LK_{10} : 2iz - x + y = 0,\\
&LK_3 : 2ix + y - z = 0, && LK_7 : 2iy + x - z = 0, && LK_{11} : 2iz + x - y = 0,\\
&LK_4 : 2ix + y + z = 0, && LK_8 : 2iy + x + z = 0. && LK_{12} : 2iz + x + y = 0.
\end{aligned}$$

Proposition 3.5

The maximal tangency lines $LK_1, LK_2, \ldots, LK_{12}$ intersect in 66 double points.

Proposition 3.6

Let $S = \mathbb{C}[x, y, z]$. Let $\mathcal{I}_3$ be the saturated ideal of the 66 points from $\mathcal{P}_2(LK)$. Then $\mathcal{I}_3$ is generated in degree 11 and its minimal free resolution has the following shape:

$$0 \longrightarrow S(-12)^{11} \xrightarrow{\alpha} S(-11)^{12} \longrightarrow S \longrightarrow S/\mathcal{I}_3 \longrightarrow 0.$$

The respective maximal tangency points $PK = \{PK_1, PK_2, \ldots, PK_{12}\}$ were also computed by Edge:

$$\begin{array}{lll} PK_1 = [i : -1 : -1], & PK_5 = [-1 : i : -1], & PK_9 = [-1 : -1 : i], \\ PK_2 = [i : -1 : 1], & PK_6 = [-1 : i : 1], & PK_{10} = [-1 : 1 : i], \\ PK_3 = [i : 1 : -1], & PK_7 = [1 : i : -1], & PK_{11} = [1 : -1 : i], \\ PK_4 = [i : 1 : 1], & PK_8 = [1 : i : 1], & PK_{12} = [1 : 1 : i]. \end{array}$$

The ideal of these points is generated in degree 4 and the three generators can be written in the following form:

$$(x^2 + y^2)(y^2 + z^2),\ (y^2 + z^2)(z^2 + x^2),\ (z^2 + x^2)(x^2 + y^2).$$

Note that these forms are similar to the forms presented earlier in Proposition 3.4. This observation leads to an interesting result: the set PK can be described as an image of the set $\mathcal{P}_2(LF)$ under the mapping

$$[x : y : z] \mapsto \left[x^2 : y^2 : z^2\right]. \tag{3.2}$$

We therefore have a specific connection between the maximal tangency points on F_4 and the maximal tangency points on K.

However, unlike the case of the Fermat quartic F_4, the set of maximal tangency points on K is not dual to the set of maximal tangency lines. We can therefore consider yet another line arrangement related to the Komiya-Kuribayashi quartic, namely

$$PK' = \mathcal{D}(PK).$$

By checking the intersection points of PK', we obtain the following result.

Proposition 3.7

The lines from the set PK' intersect in 30 double points and 6 points of multiplicity 4, i.e. we have the equality

$$t(PK') = (30, 0, 6).$$

The six points of intersection with multiplicity 4, i.e. points from the set $\mathcal{P}_4(PK')$, have the following coordinates:

$$[1 : i : 0],\ [1 : 0 : i],\ [0 : 1 : i],$$
$$[1 : -i : 0],\ [1 : 0 : -i],\ [0 : 1 : -i].$$

Furthermore, the set $\mathcal{P}_4(PK')$ is a complete intersection or curves $xyz = 0$ and the Fermat conic $F_2 : x^2 + y^2 + z^2 = 0$. It can also be described as an image of the set of maximal tangency points on F_4 under the mapping (3.2), showing yet another connection between the two quartics.

The remaining 30 points, i.e. points from the set $\mathcal{P}_2(PK')$, also form an interesting arrangement. They are all contained in the Fermat arrangement of lines, determined by linear factors of

$$(x^2 - y^2)(y^2 - z^2)(z^2 - x^2)$$

and each line contains exactly 5 of these points. It is also worth noting that the maximal tangency points $PK_1, PK_2, \ldots, PK_{12}$ also lie on these lines (with two points on each line), but none of these points are in the set $\mathcal{P}_2(PK')$.

As before, we calculate the resolution of the ideal of the considered set of points. We obtain the following result.

Proposition 3.8
Let $S = \mathbb{C}[x, y, z]$. Let $\mathcal{K}$ be the saturated ideal of the 30 points from $\mathcal{P}_2(PK')$. Then $\mathcal{K}$ is generated in degrees 6 and 7 and its minimal free resolution has the following shape:

$$0 \longrightarrow S(-9)^3 \longrightarrow S(-6) \oplus S(-7)^3 \longrightarrow S \longrightarrow S/\mathcal{K} \longrightarrow 0.$$

The obtained generator of degree 6 is naturally the union of six lines mentioned above. The remaining generators of degree 7 are all irreducible.

3.4 Sextactic points

In this section, we focus on finding the exact coordinates of the sextactic points on quartics F_4 and K.

The coordinates of the sextactic points on F_4 were already computed in the previous chapter, where we considered sextactic points on the Fermat curve F_n of degree n. In the case $n = 4$, the sextactic points have coordinates

$$\left[1 : \varepsilon_8^{2k} : \varepsilon_8^{2\ell+1}\sqrt[4]{2}\right], \ \left[\varepsilon_8^{2\ell+1}\sqrt[4]{2} : 1 : \varepsilon_8^{2k}\right], \ \left[\varepsilon_8^{2k} : \varepsilon_8^{2\ell+1}\sqrt[4]{2} : 1\right],$$

where $k, \ell \in \{0, 1, 2, 3\}$ and $\varepsilon_8 \in \mathbb{C}$ is the primitive root of unity of order 8.

Since the maximal tangency points $PK_1, \ldots, PK_{12}$ are the only inflection points on K, by Theorem 1.2 we also expect to have 48 sextactic points on K. In order to compute their exact coordinates, we can again use the method involving the second Hessian. The second Hessian of K can be computed using Singular (see Appendix, section 5.1). It turns out that in this case the equation defining the second Hessian is much more complicated than in the case of the Fermat quartic F_4. Up to a constant, it can be expressed in the following form:

$$H_2(K) = xyz(x^2 - y^2)(y^2 - z^2)(z^2 - x^2)Q(x, y, z),$$

where

$$\begin{aligned}Q(x, y, z) = {} & 1056x^{12} - 19278x^{10}y^2 - 75207x^8y^4 - 111042x^6y^6 - 75207x^4y^8 - 19278x^2y^{10} \\ & + 1056y^{12} - 19278x^{10}z^2 - 137198x^8y^2z^2 - 287194x^6y^4z^2 - 287194x^4y^6z^2 \\ & - 137198x^2y^8z^2 - 19278y^{10}z^2 - 75207x^8z^4 - 287194x^6y^2z^4 - 413110x^4y^4z^4 \\ & - 287194x^2y^6z^4 - 75207y^8z^4 - 111042x^6z^6 - 287194x^4y^2z^6 - 287194x^2y^4z^6 \\ & - 111042y^6z^6 - 75207x^4z^8 - 137198x^2y^2z^8 - 75207y^4z^8 - 19278x^2z^{10} \\ & - 19278y^2z^{10} + 1056z^{12}.\end{aligned}$$

Every point on K which also lies on $H_2(K)$ is either a maximal tangency point or a sextactic point. To find the coordinates of sextactic points, we consider all factors of $H_2(K)$.

The first three factors are x, y and z, i.e. the coordinate lines. Since all the maximal tangency points on K have non-zero coordinates, we know that the points on K lying on the coordinate lines must be the sextactic points. After some simple calculations, we obtain the following result.

Proposition 3.9

The points with coordinates

$$\begin{aligned}
SK_1 &= \left[0 : 2 : i(\sqrt{5}+1)\right], & SK_7 &= \left[0 : 2 : i(\sqrt{5}-1)\right], \\
SK_2 &= \left[0 : 2 : -i(\sqrt{5}+1)\right], & SK_8 &= \left[0 : 2 : -i(\sqrt{5}-1)\right], \\
SK_3 &= \left[i(\sqrt{5}+1) : 0 : 2\right], & SK_9 &= \left[i(\sqrt{5}-1) : 0 : 2\right], \\
SK_4 &= \left[-i(\sqrt{5}+1) : 0 : 2\right], & SK_{10} &= \left[-i(\sqrt{5}-1) : 0 : 2\right], \\
SK_5 &= \left[2 : i(\sqrt{5}+1) : 0\right], & SK_{11} &= \left[2 : i(\sqrt{5}-1) : 0\right], \\
SK_6 &= \left[2 : -i(\sqrt{5}+1) : 0\right], & SK_{12} &= \left[2 : -i(\sqrt{5}-1) : 0\right]
\end{aligned}$$

are sextactic points on the Komiya-Kuribayashi quartic.

The next three factors of $H_2(K)$, i.e. $x^2 - y^2$, $y^2 - z^2$ and $z^2 - x^2$ split into six lines, which intersect K at 24 points in total. Note that 12 of these points are actually the maximal tangency points on K. The remaining 12 points must then be the coordinates of sextactic points. Coordinates of these points are presented below.

Proposition 3.10
The points with coordinates

$$\begin{aligned} SK_{13} &= [1:1:i\sqrt{5}], & SK_{19} &= [1:-1:i\sqrt{5}], \\ SK_{14} &= [1:1:-i\sqrt{5}], & SK_{20} &= [1:-1:-i\sqrt{5}], \\ SK_{15} &= [1:i\sqrt{5}:1], & SK_{21} &= [-1:i\sqrt{5}:1], \\ SK_{16} &= [1:-i\sqrt{5}:1], & SK_{22} &= [-1:-i\sqrt{5}:1], \\ SK_{17} &= [i\sqrt{5}:1:1], & SK_{23} &= [i\sqrt{5}:1:-1], \\ SK_{18} &= [-i\sqrt{5}:1:1], & SK_{24} &= [-i\sqrt{5}:1:-1] \end{aligned}$$

are sextactic points on the Komiya-Kuribayashi quartic.

The remaining 24 sextactic points lie on a curve given by $Q(x, y, z) = 0$. It is worth noting that all 12 maximal tangency points on the Komiya-Kuribayashi K also lie on this curve. Therefore, in order to obtain the equation of a curve containing only the remaining sextactic points, we can saturate the ideal generated by Q and K by the ideal of the 12 MTPs on K. Direct computation in Singular shows that the ideal of the remaining 24 sextactic points on K is generated by the equation of K and

$$R(x, y, z) = 90x^2y^4 + 45y^6 + 124x^2y^2z^2 + 135y^4z^2 + 90x^2z^4 + 135y^2z^4 + 45z^6.$$

However, as a second generator we can also take a symmetric form, given by

$$45x^2 \cdot K(x, y, z) + R(x, y, z) = 45(x^2 + y^2 + z^2)^3 - 11x^2y^2z^2.$$

Hence we obtain the following result.

Proposition 3.11
The remaining 24 *sextactic points* $SK_{25}, \ldots, SK_{48}$ *of the Komiya-Kuribayashi quartic are the complete intersection of this quartic and a curve of degree* 6*, given by*

$$45(x^2 + y^2 + z^2)^3 - 11x^2y^2z^2 = 0.$$

3.5 Configurations of bitangent conics

In this section we are going to construct some conic arrangements, using a procedure similar to the one presented in Chapter 2. More precisely, for each quartic, we want to find all conics which are tangent to it at two sextactic points, with multiplicity 4 at both points. Then we study the arrangements of points of intersections of these conics. Again, all necessary computations were computed in Singular, using the algorithm presented in

Chapter 2.

We start with the arrangement of conics bitangent to the Fermat quartic F_4.

Proposition 3.12

There are exactly 24 *conics passing through two different sextactic points on* $F_4 : x^4 + y^4 + z^4 = 0$*, intersecting* F_4 *at these points with multiplicity* 4*. These conics are given by the following formulas:*

$$
\begin{aligned}
x^2 + (-1)^k xy + y^2 + (-1)^\ell \frac{i}{\sqrt{2}} z^2 = 0, &\qquad x^2 + (-1)^k ixy - y^2 + (-1)^\ell \frac{i}{\sqrt{2}} z^2 = 0,\\
y^2 + (-1)^k yz + z^2 + (-1)^\ell \frac{i}{\sqrt{2}} x^2 = 0, &\qquad y^2 + (-1)^k iyz - z^2 + (-1)^\ell \frac{i}{\sqrt{2}} x^2 = 0,\\
z^2 + (-1)^k zx + x^2 + (-1)^\ell \frac{i}{\sqrt{2}} y^2 = 0, &\qquad z^2 + (-1)^k izx - x^2 + (-1)^\ell \frac{i}{\sqrt{2}} y^2 = 0,
\end{aligned}
$$

for $k, \ell \in \{0, 1\}$.

Interestingly, for each sextactic point on F_4 there is exactly one such a conic passing through it. Therefore each conic connects two sextactic points, creating 24 pairs. It can be checked that the sextactic points with coordinates

$$
\left[1 : \varepsilon_8^{2k} : \varepsilon_8^{2\ell+1}\sqrt[4]{2}\right],\ \left[\varepsilon_8^{2\ell+1}\sqrt[4]{2} : 1 : \varepsilon_8^{2k}\right],\ \left[\varepsilon_8^{2k} : \varepsilon_8^{2\ell+1}\sqrt[4]{2} : 1\right]
$$

are connected with the points

$$
\left[1 : \varepsilon_8^{2k} : \varepsilon_8^{2\ell+5}\sqrt[4]{2}\right],\ \left[\varepsilon_8^{2\ell+5}\sqrt[4]{2} : 1 : \varepsilon_8^{2k}\right],\ \left[\varepsilon_8^{2k} : \varepsilon_8^{2\ell+5}\sqrt[4]{2} : 1\right],
$$

respectively, for $k, \ell \in \{0, 1, 2, 3\}$.

Next, we can take a closer look at the points at which the above bitangent conics intersect. We obtain the following result.

Proposition 3.13

The conics tangent to F_4 *at two sextactic points with multiplicity* 4 *in both points intersect in exactly* 960 *points:*

- 912 *ordinary double points,*
- 24 *points where two conics are tangent (with multiplicity* 2*),*
- 24 *points where* 4 *conics intersect (ordinary quadruple points).*

We are now going to show some properties of the obtained two sets of 24 points. It can be checked that the points from the first set (i.e. the points where two conics are tangent) have the following coordinates:

$$\begin{aligned} &\left[0 : \pm 2 : 1 + i\sqrt{3}\right], \quad \left[1 + i\sqrt{3} : 0 : \pm 2\right], \quad \left[\pm 2 : 1 + i\sqrt{3} : 0\right], \\ &\left[0 : \pm 2i : 1 + i\sqrt{3}\right], \quad \left[1 + i\sqrt{3} : 0 : \pm 2i\right], \quad \left[\pm 2i : 1 + i\sqrt{3} : 0\right], \\ &\left[0 : 1 + i\sqrt{3} : \pm 2\right], \quad \left[\pm 2 : 0 : 1 + i\sqrt{3}\right], \quad \left[1 + i\sqrt{3} : \pm 2 : 0\right], \\ &\left[0 : 1 + i\sqrt{3} : \pm 2i\right], \quad \left[\pm 2i : 0 : 1 + i\sqrt{3}\right], \quad \left[1 + i\sqrt{3} : \pm 2i : 0\right]. \end{aligned}$$

These points lie on the coordinate lines $x = 0$, $y = 0$ and $z = 0$, with eight points on each line. By computing the ideal of these points in Singular, it can be checked that they all lie on the irreducible curve of degree 8, given by

$$x^8 + y^8 + z^8 + x^4y^4 + y^4z^4 + z^4x^4 = 0$$

and therefore this set of 24 points is a complete intersection of the above curve and the curve $xyz = 0$.

It turns out that the second set of 24 intersection points (i.e. the ordinary quadruple points) has similar properties. These points have the following coordinates:

$$\begin{aligned} &\left[0 : \pm 1 : \varepsilon_8\sqrt[4]{2}\right], \quad \left[\varepsilon_8\sqrt[4]{2} : 0 : \pm 1\right], \quad \left[\pm 1 : \varepsilon_8\sqrt[4]{2} : 0\right], \\ &\left[0 : \pm i : \varepsilon_8\sqrt[4]{2}\right], \quad \left[\varepsilon_8\sqrt[4]{2} : 0 : \pm i\right], \quad \left[\pm i : \varepsilon_8\sqrt[4]{2} : 0\right], \\ &\left[0 : \varepsilon_8\sqrt[4]{2} : \pm 1\right], \quad \left[\pm 1 : 0 : \varepsilon_8\sqrt[4]{2}\right], \quad \left[\varepsilon_8\sqrt[4]{2} : \pm 1 : 0\right], \\ &\left[0 : \varepsilon_8\sqrt[4]{2} : \pm i\right], \quad \left[\pm i : 0 : \varepsilon_8\sqrt[4]{2}\right], \quad \left[\varepsilon_8\sqrt[4]{2} : \pm i : 0\right]. \end{aligned}$$

These points also lie on the coordinate lines, with eight points on each line. It is also a complete intersection of $xyz = 0$ and the irreducible curve of degree 8, given by

$$x^8 + y^8 + z^8 + \frac{5}{2}\left(x^4y^4 + y^4z^4 + z^4x^4\right) = 0$$

We can now move on to the conics bitantent to the sextactic points on the Komiya-Kuribayashi quartic K. This time, we only consider the first 24 points $SK_1, SK_2, \ldots, SK_{24}$, coordinates of which were computed in the previous section.

Proposition 3.14

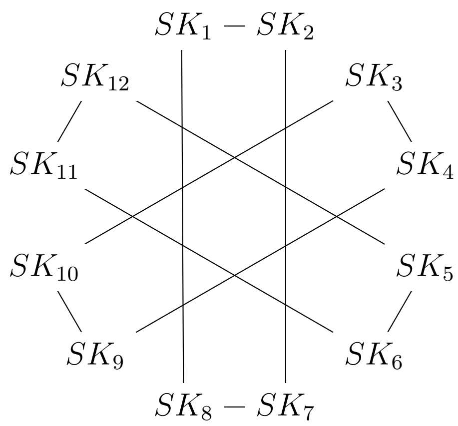
There are exactly 12 *conics, which are tangent to the Komiya-Kuribayashi quartic* K *at two sextactic points from the set* $\{SK_1, SK_2, \ldots, SK_{12}\}$*, with multiplicity* 4 *in both points. These conics are given by the following formulas:*

$$\begin{aligned} 3x^2 + 2y^2 + (-1)^k 2iyz + 2z^2 = 0, \\ 3y^2 + 2z^2 + (-1)^k 2ixz + 2x^2 = 0, \end{aligned}$$

$$
\begin{aligned}
3z^2 + 2x^2 + (-1)^k 2ixy + 2y^2 &= 0,\\
6z^2 + \left(5 + (-1)^k\sqrt{5}\right)x^2 + \left(5 + (-1)^{k+1}\sqrt{5}\right)y^2 &= 0,\\
6x^2 + \left(5 + (-1)^k\sqrt{5}\right)y^2 + \left(5 + (-1)^{k+1}\sqrt{5}\right)z^2 &= 0,\\
6y^2 + \left(5 + (-1)^k\sqrt{5}\right)z^2 + \left(5 + (-1)^{k+1}\sqrt{5}\right)x^2 &= 0,
\end{aligned}
$$

for $k \in \{0, 1\}$.

In this case, it can be checked that for each point from the set $\{SK_1, \ldots, SK_{12}\}$, there are exactly two conics passing through it. Therefore each sextactic point from this set is connected with two other points. All these connections are shown on the graph below.



Furthermore, it can be checked that apart from the points from the set $\{SK_1, SK_2, \ldots, SK_{12}\}$, these 12 conics intersect only in ordinary double points.

We can now move on to the next subset of sextactic points on K.

Proposition 3.15
There are exactly 6 *conics, which are tangent to the Komiya-Kuribayashi quartic* K *at two sextactic points from the set* $\{SK_{13}, SK_{14}, \ldots, SK_{24}\}$*, with multiplicity* 4 *in both points. These conics are given by the following formulas:*

$$
\begin{aligned}
4x^2 + 11y^2 + (-1)^k 2iyz + 11z^2 &= 0,\\
4y^2 + 11z^2 + (-1)^k 2ixz + 11x^2 &= 0,\\
4z^2 + 11x^2 + (-1)^k 2ixy + 11y^2 &= 0.
\end{aligned}
$$

for $k \in \{0, 1\}$.

In this case, just like in the case of sextactic points on F_4, there is only one conic passing through each point from the set $\{SK_{13}, SK_{14}, \ldots, SK_{24}\}$ and therefore these points are connected into pairs. More precisely, the point SK_{2k-1} is connected with SK_{2k} for

$k \in \{7, \ldots, 12\}$. Furthermore, it can be checked that these 6 conics intersect only in 60 ordinary double points.

Remark 3.1

It turns out that these two sets $\{SK_1, \ldots, SK_{12}\}$ and $\{SK_{13}, \ldots, SK_{24}\}$ of sextactic points on K cannot be connected by a bitangent conic – there is no conic tangent to the Komiya-Kuribayashi quartic K at one point from the first set and at another point from the second set, with multiplicity 4 in both points.

Chapter 4

Free arrangements of three conics

In this chapter, we give a complete classification of free arrangements of three smooth conics in $\mathbb{P}^2(\mathbb{C})$ with ADE singularities, up to projective equivalence. Initial results of this chapter were announced in [MZZ25]. This chapter provides significantly more detailed proofs of all propositions, includes additional examples, and lists the coordinates of all singularities in the arrangements.

4.1 Preliminaries

Definition 4.1

Let p be an isolated singularity of a curve given by a polynomial $f \in \mathbb{C}[x, y]$. By changing the local coordinates, we can assume $p = (0, 0)$.

1. The number
$$\mu_p = \dim_{\mathbb{C}} \left(\mathbb{C}[[x, y]] / \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \right)$$
is called *the Milnor number* of f at p.

2. The number
$$\tau_p = \dim_{\mathbb{C}} \left(\mathbb{C}[[x, y]] / \left\langle f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \right)$$
is called *the Tjurina number* of f at p.

For a given curve $C \subset \mathbb{P}^2(\mathbb{C})$, we also define the *total Tjurina number* of C as

$$\tau(C) = \sum_{p \in \mathrm{Sing}(C)} \tau_p.$$

Now, let $S = \mathbb{C}[x, y, z]$ be the coordinate ring of $\mathbb{P}^2(\mathbb{C})$. Let $C : f = 0$ be a reduced curve in $\mathbb{P}^2(\mathbb{C})$ of degree d, defined by $f \in S$. We consider the graded S-module $D_0(f)$ of

Jacobian syzygies of f, defined as

$$D_0(f) = \{(a, b, c) \in S^3 : af_x + bf_y + cf_z = 0\}.$$

We denote by $\mathrm{mdr}(f) = d_1$ the minimal degree of a nontrivial syzygy in $D_0(f)$.

A useful criterion for checking if a given plane curve C is free is the Du Plessis–Wall Theorem (see [dPW99]), which gives a necessary and sufficient condition for freeness, based on the total Tjurina number of C.

Theorem 4.1 (Du Plessis–Wall Theorem)
A reduced plane curve $C : f = 0$ with $\mathrm{mdr}(f) \leq \frac{d-1}{2}$ is free if and only if

$$\tau(C) = (d-1)^2 - d_1(d - d_1 - 1).$$

We are going to study arrangements of three smooth conics in $\mathbb{P}^2(\mathbb{C})$, with only ADE singularities. The list of all possible ADE singularities in such configurations is presented below, along with their brief descriptions:

- A_1 – two conics intersect transversally (a node),
- A_3 – two conics are tangent with multiplicity 2 (a tacnode),
- A_5 – two conics are tangent with multiplicity 3,
- A_7 – two conics are tangent with multiplicity 4,
- D_4 – three conics intersect pairwise transversally,
- D_6 – two conics form the A_3 singularity and the third one passes pairwise transversally through it,
- D_8 – two conics form the A_5 singularity and the third one passes pairwise transversally through it,
- D_{10} – two conics form the A_7 singularity and the third one passes pairwise transversally through it.

Below we also present the local normal forms of these singularities, based on the equations from Arnold's paper [Arn76]:

$$\begin{array}{ll} A_k \text{ with } k \geq 1 : & x^2 + y^{k+1} = 0, \\ D_k \text{ with } k \geq 4 : & y^2 x + x^{k-1} = 0. \end{array}$$

Table 4.1: Local normal forms.

Each arrangement of three smooth plane conics can be split into three arrangements of pairs of conics (by considering each pair of conics separately). There are only 5 possible configurations of two smooth conics (based on their intersection patterns) and they are all shown in Table 4.2. In the first column we present a graph notation (based on the notation from [Sar10]), which we are going to use to represent the structure of the arrangements of three conics. In the last column, we present our own symbolic notation for each pair – this notation will be used to write down the decomposition into pairs (see Example 4.2 below).

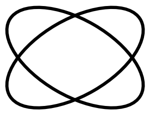
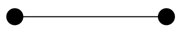
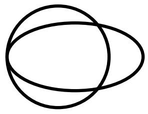
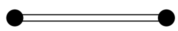
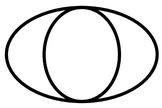
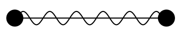
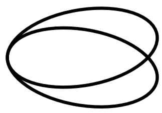
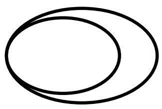
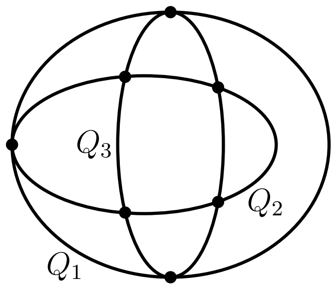
Graph notation	Picture	Singularities	Symbolic notation
 Q_1 Q_2		four A_1	$\boldsymbol{\mathcal{N}}$
 Q_1 Q_2		one A_3, two A_1	$\boldsymbol{\mathcal{T}}$
 Q_1 Q_2		two A_3	$\boldsymbol{\mathcal{T}}^{\mathbf{2}}$
 Q_1 Q_2		one A_5, one A_1	$\boldsymbol{\mathcal{A}}^{\mathbf{5}}$
 Q_1 Q_2		one A_7	$\boldsymbol{\mathcal{A}}^{\mathbf{7}}$

Table 4.2: Configurations of two smooth conics.

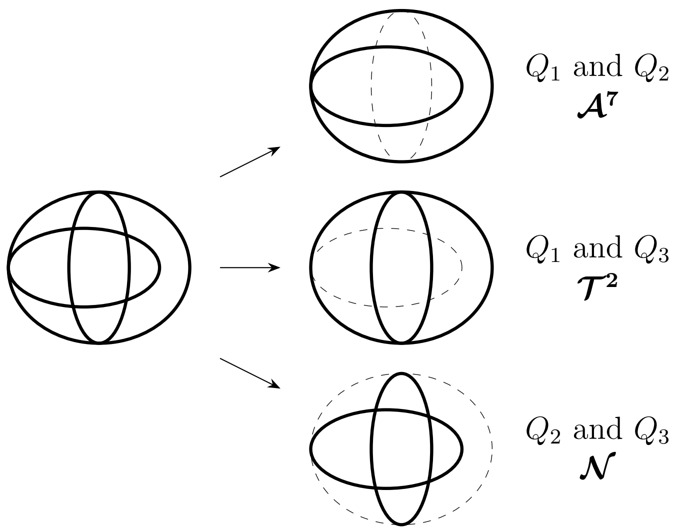
Example 4.2
Consider the following arrangement of three smooth plane conics:

$$\begin{aligned} Q_1 &: x^2 - yz = 0, \\ Q_2 &: x^2 + z^2 - yz = 0, \\ Q_3 &: 2x^2 - y^2 - z^2 = 0. \end{aligned}$$

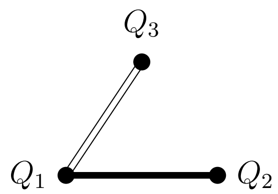
This arrangement is illustrated in the picture below:



By splitting this arrangement into three pairs of conics, we obtain the following:



Therefore the decomposition into pairs in this case can be written as $\boldsymbol{\mathcal{A}^7} + \boldsymbol{\mathcal{T}^2} + \boldsymbol{\mathcal{N}}$. Additionally, this arrangement can be represented by the following graph:



If a configuration of three conics can be split into two or more pairs of the same type, we write the decomposition in the shorter form. For instance, instead of $\boldsymbol{\mathcal{A}^7} + \boldsymbol{\mathcal{T}^2} + \boldsymbol{\mathcal{T}^2}$, we write $\boldsymbol{\mathcal{A}^7} + 2 \cdot \boldsymbol{\mathcal{T}^2}$.

Table 4.3 shows some additional information about the above singularities, namely the values of the Milnor number μ_p (which is in this case also equal to the Tjurina number) and the notation that we are going to use to describe the weak combinatorics of the conic arrangements. We write each weak combinatoric as a vector. For example

$$(n_2, t_3, n_3, t_5, d_6, t_7, d_8, d_{10}) = (2, 1, 0, 0, 0, 2, 0, 0)$$

describes an arrangement with two nodes (A_1 singularites), one tacnode (A_3 singularity) and two A_7 singularities.

Singularity	μ_p	Number of occurences
A_1	1	n_2
A_3	3	t_3
D_4	4	n_3
A_5	5	t_5
D_6	6	d_6
A_7	7	t_7
D_8	8	d_8
D_{10}	10	d_{10}

Table 4.3: ADE singularities.

4.2 Basic tools

The following two propositions, based on the results from [Sar10], will be one of the main tools used later in our constructions. The parametrizations included in both propositions can also be found in [Sar10].

Proposition 4.3 (Proposition 4.1.1 [Sar10])
Any configuration of two smooth conics in $\mathbb{P}^2(\mathbb{C})$ with a singularity of type A_7 is projectively equivalent to the conics

$$Q_1 : x^2 - yz = 0,$$
$$Q_2 : x^2 + az^2 - yz = 0,$$

with $a \in \mathbb{C} \setminus \{0\}$. These conics have the following parametrizations:

$$Q_1 = \left\{[uv : v^2 : u^2],\ [u : v] \in \mathbb{P}^1(\mathbb{C})\right\},$$
$$Q_2 = \left\{[st : as^2 + t^2 : s^2],\ [s : t] \in \mathbb{P}^1(\mathbb{C})\right\}.$$

Proposition 4.4 (Proposition 4.2.1 [Sar10])
Any configuration of two smooth conics in $\mathbb{P}^2(\mathbb{C})$ with a singularity of type A_5 is projectively equivalent to the conics

$$Q_1 : x^2 - yz = 0,$$
$$Q_2 : x^2 + by^2 + cxy - yz = 0,$$

with $b, c \in \mathbb{C}$ and $c \neq 0$. These conics have the following parametrizations:

$$Q_1 = \left\{[uv : v^2 : u^2],\ [u : v] \in \mathbb{P}^1(\mathbb{C})\right\},$$
$$Q_2 = \left\{[st : s^2 : t^2 + bs^2 + cst],\ [s : t] \in \mathbb{P}^1(\mathbb{C})\right\}.$$

Starting from the equations of Q_1 and Q_2 either from Proposition 4.3 or Proposition 4.4, we try to find the equation of Q_3 based on the intersection patterns with the first two conics. In order to find it, we use the following fact.

Fact 4.1
Let Q_1 and Q_2 be two distinct smooth conics.

a) If Q_1 and Q_2 form the $\boldsymbol{\mathcal{T}}$ arrangement (i.e. with one A_3 singularity and two nodes), then the equation of Q_2 can be written in the form

$$Q_2 : \lambda Q_1 + \ell_1 \ell_2 = 0,$$

where $\lambda \in \mathbb{C} \setminus \{0\}$, ℓ_1 is the equation of a line tangent to Q_1 at the A_3 singularity and ℓ_2 is the equation of a line passing through both nodes.

b) If Q_1 and Q_2 form the $\boldsymbol{\mathcal{T}^2}$ arrangement (i.e. with two A_3 singularities), then the equation of Q_2 can be written in the form

$$Q_2 : \lambda Q_1 + \ell_1 \ell_2 = 0,$$

where $\lambda \in \mathbb{C} \setminus \{0\}$ and ℓ_1, ℓ_2 are the equation of lines tangent to Q_1 at the respective A_3 singularities.

c) If Q_1 and Q_2 form the $\boldsymbol{\mathcal{A}^5}$ arrangement (i.e. with singularities A_5 and A_1), then the equation of Q_2 can be written in the form

$$Q_2 : \lambda Q_1 + \ell_1 \ell_2 = 0,$$

where $\lambda \in \mathbb{C} \setminus \{0\}$, ℓ_1 is the equation of a line tangent to Q_1 at the A_5 singularity and ℓ_2 is the equation of a line passing through both singularities.

d) If Q_1 and Q_2 form the $\boldsymbol{\mathcal{A}^7}$ arrangement (i.e. with one A_7 singularity), then the equation of Q_2 can be written in the form

$$Q_2 : \lambda Q_1 + \ell^2 = 0,$$

where $\lambda \in \mathbb{C} \setminus \{0\}$ and ℓ is the equation of a line tangent to Q_1 at the A_7 singularity.

4.3 Classification of free arrangements

In this section, we determine all theoretically possible weak combinatorics of free arrangements of three smooth conics in $\mathbb{P}^2(\mathbb{C})$. Then we verify which of these weak combinatorics can actually be realized geometrically.

First, we recall some additional definitions, based on [Pok24] and [DS14]. The germ (C, p) is *weighted homogeneous* of type $(w_1, w_2; 1)$ with $0 < w_1, w_2 \leq \frac{1}{2}$ if there are local analytic coordinates y_1, y_2 centered at $p = (0, 0)$ and a polynomial

$$g(y_1, y_2) = \sum_{u,v} c_{u,v} y_1^u y_2^v$$

with $c_{u,v} \in \mathbb{C}$, where the sum is over all pairs $(u, v) \in \mathbb{N}^2$ such that $uw_1 + vw_2 = 1$. With this description, the *Arnold exponent* (a log canonical threshold) of p can be defined as

$$\alpha_p = w_1 + w_2.$$

We also define

$$\alpha_C = \min\{\alpha_p : p \in \mathrm{Sing}(C)\}.$$

If C is a configuration of three smooth conics with only ADE singularities, then by [DS14, Theorem 2.1] we have $d_1 \geq 6\alpha_C - 2$. On the other hand, since our configuration has only ADE singularities, by [Dim87, Theorem 7.45] we have $\alpha_C > \frac{1}{2}$ and therefore $d_1 > 6 \cdot \frac{1}{2} - 2 = 1$. However, since $d_1 < \frac{d}{2}$ for free curves of degree d, in our case we also have $d_1 < 3$. Hence $d_1 = 2$. Therefore, by Theorem 4.1, we obtain

$$\tau(C) = (6-1)^2 - 2(6-2-1) = 19.$$

Furthermore, since C consists of three conics, by adding the multiplicities of all singularities of C, we need to obtain $\binom{4}{2} = 12$. Based on these two informations, if $(n_2, t_3, n_4, t_5, d_6, t_7, d_8, d_{10})$ is a weak combinatoric of a free configuration of three smooth conics, then it satisfies the following system of equations:

$$\begin{cases} n_2 + 3t_3 + 4n_3 + 5t_5 + 6d_6 + 7t_7 + 8d_8 + 10d_{10} = 19, \\ n_2 + 2t_3 + 3n_3 + 3t_5 + 4d_6 + 4t_7 + 5d_8 + 6d_{10} = 12. \end{cases}$$

By finding all solutions of the above system of equations (in nonnegative integers), we obtain the following result.

Theorem 4.5

A configuration of three smooth conics with only ADE *singularities is free if and only if it has one of the following weak combinatorics:*

$$(n_2, t_3, n_3, t_5, d_6, t_7, d_8, d_{10}) \in \{(0,3,0,2,0,0,0,0), (1,1,0,3,0,0,0,0),$$
$$(0,0,1,3,0,0,0,0), (0,1,0,2,1,0,0,0), (0,4,0,0,0,1,0,0), (1,2,0,1,0,1,0,0),$$

$$(0,1,1,1,0,1,0,0),(2,0,0,2,0,1,0,0),(0,2,0,0,1,1,0,0),(1,0,0,1,1,1,0,0),$$
$$(0,0,0,0,2,1,0,0),(2,1,0,0,0,2,0,0),(1,0,1,0,0,2,0,0),(0,2,0,1,0,0,1,0),$$
$$(1,0,0,2,0,0,1,0),(0,0,0,1,1,0,1,0),(1,1,0,0,0,1,1,0),(0,0,1,0,0,1,1,0),$$
$$(0,1,0,0,0,0,2,0),(0,3,0,0,0,0,0,1),(1,1,0,1,0,0,0,1),(0,0,1,1,0,0,0,1),$$
$$(0,1,0,0,1,0,0,1),(2,0,0,0,0,1,0,1),(1,0,0,0,0,0,1,1)\}.$$

Next, we want to verify which of these weak combinatorics can be realized over $\mathbb{C}$ by three smooth conics. It turns out that many of these combinatorics actually cannot be realized and it can be proved using simple observations. Two examples are provided below.

Example 4.6
Consider the weak combinatoric $(n_2, t_3, n_3, t_5, d_6, t_7, d_8, d_{10}) = (1,1,0,3,0,0,0,0)$. Since it has three A_5 singularities, it's decomposition into pairs can only be $3 \cdot \boldsymbol{\mathcal{A}^5}$. However, a configuration with such a decomposition does not have any A_3 singularities, which leads to a contradiction.

Example 4.7
Consider the weak combinatoric $(n_2, t_3, n_3, t_5, d_6, t_7, d_8, d_{10}) = (0,2,0,0,1,1,0,0)$. Since it has an A_7 singularity, one of the pairs in the decomposition is $\boldsymbol{\mathcal{A}^7}$ and the conics from this pair does not intersect in any other point. However, in this combinatoric we also have one D_6 singularity, in which all three conics intersect. This is a contradiction.

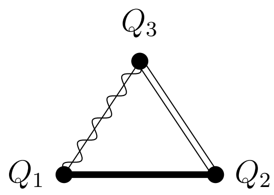
All the technical lemmas used to exclude other simpler cases from the list are described in more detail in [MZZ25]. After this whole process, we are left with the following list of 9 combinatorics:

$$(n_2, t_3, n_3, t_5, d_6, t_7, d_8, d_{10}) \in \{(0,0,1,3,0,0,0,0),$$
$$(0,1,0,2,1,0,0,0),(0,4,0,0,0,1,0,0),(1,2,0,1,0,1,0,0),$$
$$(2,0,0,2,0,1,0,0),(2,1,0,0,0,2,0,0),(1,0,0,2,0,0,1,0),$$
$$(0,1,0,0,0,0,2,0),(1,1,0,1,0,0,0,1)\}.$$

These cases are considered separately below. We start with three weak combinatorics that cannot be realized geometrically over $\mathbb{C}$.

Proposition 4.8
The weak combinatoric $(n_2, t_3, n_3, t_5, d_6, t_7, d_8, d_{10}) = (1,2,0,1,0,1,0,0)$ *cannot be realized over* $\mathbb{C}$ *by smooth conics.*

Proof. For this arrangement, the only possible decomposition into pairs is $\boldsymbol{\mathcal{T}^2} + \boldsymbol{\mathcal{A}^5} + \boldsymbol{\mathcal{A}^7}$. Suppose that such a configuration of three conics Q_1, Q_2 and Q_3 exists over $\mathbb{C}$ and it is represented by the following graph:



By Proposition 4.3 we may assume that

$$Q_1 : x^2 - yz = 0, \quad Q_2 : x^2 + az^2 - yz = 0$$

for some $a \in \mathbb{C} \setminus \{0\}$. These two conics intersect with multiplicity 4 at the point $[0 : 1 : 0]$. Based on the parametrization of Q_1 from Proposition 4.3, the conic Q_3 intersects with Q_1 at two points $[p : p^2 : 1]$ and $[q : q^2 : 1]$ with multiplicities 3 and 1, respectively, for some $p, q \in \mathbb{C}$, $p \neq q$. The line tangent to Q_1 at $[p : p^2 : 1]$ is given by $2px - y - p^2z = 0$, while the line passing through $[p : p^2 : 1]$ and $[q : q^2 : 1]$ is given by $(p+q)x - y - pqz = 0$. Therefore by Fact 4.1 c) the equation of Q_3 can be written as

$$\begin{aligned} Q_3 &: \lambda(x^2 - yz) + (2px - y - p^2z)\big((p+q)x - y - pqz\big) \\ &= (\lambda + 2p^2 + 2pq)x^2 + y^2 + p^3qz^2 - (3p+q)xy \\ &- (p^3 + 3p^2q)xz + (p^2 + pq - \lambda)yz = 0 \end{aligned}$$

for some $\lambda \in \mathbb{C} \setminus \{0\}$. By substituting the affine parametrization of Q_2 (which is $x = t$, $y = t^2 + a$ and $z = 1$) into the above equation, we obtain

$$\begin{aligned} f(t) = t^4 &- (3p+q)t^3 + (3p^2 + 3pq + 2a)t^2 \\ &- (p^3 + 3p^2q + 3ap + aq)t + p^3q + a^2 + ap^2 + apq - \lambda a = 0. \end{aligned}$$

On the other hand, since Q_2 and Q_3 intersect in two points with multiplicity 2, we know that $f(t)$ can also be written in the form

$$\begin{aligned} f(t) = (t-m)^2(t-n)^2 = t^4 - (2m+2n)t^3 + (m^2 + 4mn + n^2)t^2 \\ - (2m^2n + 2mn^2)t + m^2n^2 = 0. \end{aligned}$$

Then, by comparing the coefficients in both obtained equations of $f(t)$, we obtain the following system of equations:

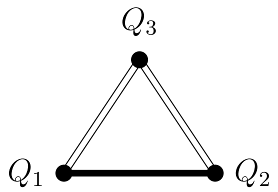
$$\begin{cases} 3p + q = 2m + 2n, & \text{(I)} \\ 3p^2 + 3pq + 2a = m^2 + 4mn + n^2, & \text{(II)} \\ p^3 + 3p^2q + 3ap + aq = 2m^2n + 2mn^2, & \text{(III)} \\ p^3q + a^2 + ap^2 + apq - \lambda a = m^2n^2. & \text{(IV)} \end{cases}$$

From equations (I) and (II) we obtain $m+n = \frac{3}{2}p + \frac{1}{2}q$ and $mn = \frac{3}{8}p^2 + \frac{3}{4}pq - \frac{1}{8}q^2 + a$. Then, after substituting these values into equation (III) and simplifying, we obtain $(p-q)^3 = 0$, which implies $p = q$, contradicting our assumptions. ■

Proposition 4.9

The weak combinatoric $(n_2, t_3, n_3, t_5, d_6, t_7, d_8, d_{10}) = (0, 4, 0, 0, 0, 1, 0, 0)$ *cannot be realized over* $\mathbb{C}$ *by smooth conics.*

Proof. For this arrangement, the only possible decomposition into pairs is $\boldsymbol{\mathcal{A}}^{\mathbf{7}} + 2 \cdot \boldsymbol{\mathcal{T}}^{\mathbf{2}}$. Suppose that such a configuration of three conics Q_1, Q_2 and Q_3 exists over $\mathbb{C}$ and it is represented by the following graph:



By Proposition 4.3 we may assume that

$$Q_1 : x^2 - yz = 0, \quad Q_2 : x^2 + az^2 - yz = 0$$

for some $a \in \mathbb{C} \setminus \{0\}$. These two conics intersect with multiplicity 4 at the point $[0:1:0]$. Based on the parametrization of Q_1 from Proposition 4.3, the conic Q_1 intersects with Q_3 at two points $[p : p^2 : 1]$ and $[q : q^2 : 1]$, with multiplicities 2 in both points, for some $p, q \in \mathbb{C}$, $p \neq q$. The lines tangent to Q_1 at these points are given by $2px - y - p^2z = 0$ and $2qx - y - q^2z = 0$. Therefore by Fact 4.1 b) the equation of Q_3 can be written as

$$\begin{aligned} Q_3 : \lambda(x^2 - yz) &+ (2px - y - p^2z)(2qx - y - q^2z) \\ &= (\lambda + 4pq)x^2 + y^2 + p^2q^2z^2 - (2p+2q)xy \\ &- (2p^2q + 2pq^2)xz + (p^2 + q^2 - \lambda)yz = 0 \end{aligned}$$

with $\lambda \in \mathbb{C} \setminus \{0\}$.

On the other hand, based on the parametrization of Q_2 from Proposition 4.3, the conic Q_2 intersects with Q_3 at two points $[r : a + r^2 : 1]$ and $[s : a + s^2 : 1]$, with multiplicities 2 in both points, for some $r, s \in \mathbb{C}$, $r \neq s$. The lines tangent to Q_2 at these two points are given by equations $2rx - y + (a - r^2)z = 0$ and $2sx - y + (a - s^2)z = 0$. Hence by Fact

4.1 b) the equation of Q_3 can also be written as

$$\begin{aligned} Q_3 &: \mu(x^2 + az^2 - yz) + \Big(2rx - y + (a - r^2)z\Big)\Big(2sx - y + (a - s^2)z\Big) \\ &= (\mu + 4rs)x^2 + y^2 + (\mu a + a^2 - ar^2 - as^2 + r^2s^2)z^2 - (2r + 2s)xy \\ &- (2r^2s + 2rs^2 - 2ar - 2as)xz + (r^2 + s^2 - \mu - 2a)yz = 0 \end{aligned} \tag{4.1}$$

Therefore, by comparing the coefficients in both equations, we obtain

$$\begin{cases} \lambda + 4pq = \mu + 4rs, & \text{(I)} \\ p^2q^2 = \mu a + a^2 - ar^2 - as^2 + r^2s^2, & \text{(II)} \\ p + q = r + s, & \text{(III)} \\ p^2q + pq^2 = r^2s + rs^2 - ar - as, & \text{(IV)} \\ p^2 + q^2 - \lambda = r^2 + s^2 - \mu - 2a. & \text{(V)} \end{cases}$$

From equations (I), (III) and (V) (in (III) we square both sides), we obtain $pq = rs - a$. Then, by substituting it into the equation (II) and simplifying, we get $\mu = (r - s)^2$. However, after substituting the obtained value of μ into equation (4.1), we obtain

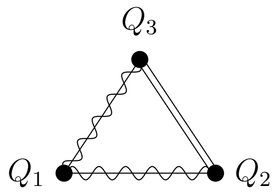
$$\begin{aligned} Q_3 &: (r + s)^2x^2 + y^2 + (a - rs)^2z^2 - 2(r + s)xy + 2(a - rs)(r + s)xz - 2(a - rs)yz \\ &= \Big((r + s)x - y + (a - rs)z\Big)^2 = 0, \end{aligned}$$

which is a contradiction with irreducibility of Q_3. ■

Proposition 4.10

The weak combinatoric $(n_2, t_3, n_3, t_5, d_6, t_7, d_8, d_{10}) = (0, 1, 0, 2, 1, 0, 0, 0)$ *cannot be realized over* $\mathbb{C}$ *by smooth conics.*

Proof. For this arrangement, the only possible decomposition into pairs is $2 \cdot \boldsymbol{\mathcal{A}^5} + \boldsymbol{\mathcal{T}^2}$. Assume that this configuration of three conics is represented by the following graph:



By Proposition 4.4 we may assume that

$$Q_1 : x^2 - yz = 0, \quad Q_2 : x^2 + by^2 + cxy - yz = 0$$

for some $b, c \in \mathbb{C}$ and $c \neq 0$. These conics intersect at points $[0 : 0 : 1]$ and $[-bc : c^2 : b^2]$, with respective multiplicities 3 and 1. The second point is the D_6 singularity in our configuration.

The conics Q_1 and Q_3 also intersect in two points with multiplicities 3 and 1. The point in which these conics intersect transversally must be the D_6 singularity $[-bc : c^2 : b^2]$. Based on the parametrization of Q_1 from Proposition 4.4, Q_3 intersects with Q_1 with multiplicity 3 at point $[p : 1 : p^2]$ for some $p \in \mathbb{C}$ and $p \neq -\frac{b}{c}$ (in order not to coincide with the D_6 singularity). The line tangent to Q_1 at this point is given by $2px - p^2y - z = 0$, while the line passing through $[p : 1 : p^2]$ and $[-bc : c^2 : b^2]$ is given by $(b - cp)x - bpy + cz = 0$. Hence by Fact 4.1 c) the equation of Q_3 can be written as

$$
\begin{aligned}
Q_3 : \lambda(x^2 - yz) &+ (2px - p^2y - z)\big((b - cp)x - bpy + cz\big) \\
&= (\lambda + 2bp - 2cp^2)x^2 + bp^3y^2 - cz^2 + (cp^3 - 3bp^2)xy \\
&+ (3cp - b)xz + (bp - cp^2 - \lambda)yz = 0
\end{aligned}
$$

for some $\lambda \in \mathbb{C} \setminus \{0\}$.

On the other hand, the conics Q_2 and Q_3 intersect at $[-bc : c^2 : b^2]$ with multiplicity 2. Based on the parametrization of Q_2 from Proposition 4.4, the other point of intersection, also with multiplicity 2, has coordinates $[q : 1 : q^2 + b + cq]$ for some $q \in \mathbb{C}$, $q \neq -\frac{b}{c}$. The lines tangent to Q_2 at both points are given by $(c^3 - 2bc)x + (bc^2 - b^2)y - c^2z = 0$ and $(c + 2q)x + (b - q^2)y - z = 0$. Therefore, by Fact 4.1 b), the equation of Q_3 can also be written as

$$
\begin{aligned}
Q_3 : \mu(x^2 + by^2 + cxy - yz) &+ \big((c^3 - 2bc)x + (bc^2 - b^2)y - c^2z\big)\big((c + 2q)x + (b - q^2)y - z\big) \\
&= (\mu + c^4 + 2c^3q - 2bc^2 - 4bcq)x^2 + (\mu b + b^2c^2 - bc^2q^2 - b^3 + b^2q^2)y^2 + c^2z^2 \\
&+ (\mu c + 2bc^3 - c^3q^2 - 3b^2c + 2bc^2q + 2bcq^2 - 2b^2q)xy + (2bc - 2c^3 - 2c^2q)xz \\
&+ (b^2 - 2bc^2 + c^2q^2 - \mu)yz = 0
\end{aligned}
$$

We can now compare the coefficients in both equations (after multiplying the first one by $(-c)$ to match the coefficients at z^2). We obtain the following system of equations:

$$
\begin{cases}
-\lambda c - 2bcp + 2c^2p^2 = \mu + c^4 + 2c^3q - 2bc^2 - 4bcq, & \text{(I)} \\
-bcp^3 = \mu b + b^2c^2 - bc^2q^2 - b^3 + b^2q^2, & \text{(II)} \\
-c^2p^3 + 3bcp^2 = \mu c + 2bc^3 - c^3q^2 - 3b^2c + 2bc^2q + 2bcq^2 - 2b^2q, & \text{(III)} \\
-3c^2p + bc = 2bc - 2c^3 - 2c^2q, & \text{(IV)} \\
-bcp + c^2p^2 + \lambda c = b^2 - 2bc^2 + c^2q^2 - \mu. & \text{(V)}
\end{cases}
$$

From (IV) we get $b = 2c^2 - 3cp + 2cq$ and then from (V) we obtain $\mu = 5c^2p^2 - 10c^2pq +$

$5c^2q^2 - 4c^3p + 4c^3q - \lambda c$. Then, after substituting these values into equation (I) and factorizing, we obtain

$$3c^2(c - p + q)^2 = 0.$$

and hence $c = p - q$ (because $c \neq 0$). After all these substitutions the equation (III) can be reduced to $\lambda(p-q)^2 = 0$, hence $p - q = 0$ (because $\lambda \neq 0$). But this also indicates that $c = 0$, which is a contradiction. ■

It turns out that the remaining 6 combinatorics actually can be realized over $\mathbb{C}$ by a configuration of three smooth conics. Again, we consider each case separately. For each weak combinatoric, we give the equations defining all three conics (up to a projective equivalence) as well as the coordinates of all singularities.

Proposition 4.11

Any free configuration of three smooth conics with weak combinatorics

$$(n_2, t_3, n_3, t_5, d_6, t_7, d_8, d_{10}) = (2, 0, 0, 2, 0, 1, 0, 0)$$

is projectively equivalent to the conics

$$\begin{aligned}
&Q_1 : x^2 - yz = 0,\\
&Q_2 : x^2 + az^2 - yz = 0,\\
&Q_3 : \left(\frac{(m+p)(m-p)^3}{a} + \frac{a(m+p)}{m-p} - \frac{2}{3}(m^2 - 8mp + p^2)\right)x^2 + y^2\\
&\quad + \left(\frac{2ap^3}{3(m-p)} + 2mp^3 - p^4\right)z^2 - \left(\frac{2a}{3(m-p)} + 2m + 2p\right)xy\\
&\quad - \left(\frac{2ap^2}{m-p} + 6mp^2 - 2p^3\right)xz - \left(\frac{(m+p)(m-p)^3}{a} + a - \frac{2}{3}(m^2 + mp + p^2)\right)yz = 0,
\end{aligned}$$

with $a, m, p \in \mathbb{C}$, $m \neq p$ and $a^2 = -3(m-p)^4$. In this arrangement, the singularities have the following coordinates:

- $[0 : 1 : 0]$ – *an A_7 singularity, intersection of Q_1 and Q_2,*
- $[p : p^2 : 1]$ – *an A_5 singularity, intersection of Q_1 and Q_3,*
- $[m : m^2 + a : 1]$ – *an A_5 singularity, intersection of Q_2 and Q_3,*
- $[q : q^2 : 1]$ – *an A_1 singularity, intersection of Q_1 and Q_3,*
- $[n : n^2 + a : 1]$ – *an A_1 singularity, intersection of Q_2 and Q_3,*

with $q = 2m - p + \frac{2a}{3(m-p)}$ and $n = 2p - m + \frac{2a}{3(m-p)}$.

The statement and the proof of the above proposition was already published in [Sar10] (Proposition 4.2.5). and the coordinates of singularities can be extracted from the proof. However, the formula for Q_3 presented there contains mistakes. The above formula is the correct one.

Since the parameters a, m and p have to satisfy the equality $a^2 = -3(m-p)^4$ with $m \neq p$ (which implies $a^2 < 0$), this arrangement cannot be realized over $\mathbb{R}$.

Proposition 4.12

Any free configuration of three smooth conics with weak combinatorics

$$(n_2, t_3, n_3, t_5, d_6, t_7, d_8, d_{10}) = (0, 0, 1, 3, 0, 0, 0, 0)$$

is projectively equivalent to the conics

$$\begin{aligned}
&Q_1 : x^2 - yz = 0,\\
&Q_2 : x^2 + by^2 + cxy - yz = 0,\\
&Q_3 : c^2(7c^4 - 54bc^2 + 108b^2)x^2 - b(c^2-3b)^3y^2 + 27c^4z^2 - c(c^2-12b)(c^2-3b)^2xy\\
&\quad - 27c^3(c^2-4b)xz + c^2(2c^4 - 27bc^2 + 54b^2)yz = 0
\end{aligned}$$

with $b, c \in \mathbb{C}$ and $c \neq 0$. In this arrangement, the singularities have the following coordinates:

- $[0:0:1]$ – *an A_5 singularity, intersection of Q_1 and Q_2,*
- $\left[-bc + \frac{1}{3}c^3 : c^2 : (b - \frac{1}{3}c^2)^2\right]$ – *an A_5 singularity, intersection of Q_1 and Q_3,*
- $\left[-bc - \frac{1}{3}c^3 : c^2 : (b + \frac{1}{3}c^2)^2 - \frac{1}{3}c^4\right]$ – *an A_5 singularity, intersection of Q_2 and Q_3,*
- $[-bc : c^2 : b^2]$ – *a D_4 singularity.*

Proof. For this arrangement, the only possible decomposition into pairs is $3 \cdot \boldsymbol{\mathcal{A}^5}$. By Proposition 4.4 we may assume that

$$Q_1 : x^2 - yz = 0, \quad Q_2 : x^2 + by^2 + cxy - yz = 0$$

for some $b, c \in \mathbb{C}$ and $c \neq 0$. These conics intersect at points $[0:0:1]$ and $[-bc : c^2 : b^2]$ with multiplicities 3 and 1, respectively, and the second point is the D_4 singularity in our configuration.

Based on the parametrization of Q_1 from Proposition 4.4 we know that Q_3 intersect with Q_1 with multiplicity 3 at point $[p : 1 : p^2]$ for some $p \in \mathbb{C}$ and $p \neq -\frac{b}{c}$ (otherwise it coincides with $[-bc : c^2 : b^2]$). The line tangent to Q_1 at this point is given by $2px - p^2y - z = 0$, while the line passing through $[p : 1 : p^2]$ and $[-bc : c^2 : b^2]$ is given by

$(b-cp)x - bpy + cz = 0$. Therefore by Fact 4.1 c) we get that the equation of Q_3 can be written as

$$\begin{aligned} Q_3 : \lambda(x^2 - yz) + (2px - p^2y - z)\big((b-cp)x - bpy + cz\big) \\ = (\lambda + 2bp - 2cp^2)x^2 + bp^3y^2 - cz^2 + (cp^3 - 3bp^2)xy \\ + (3cp - b)xz + (bp - cp^2 - \lambda)yz = 0 \end{aligned}$$

for some $\lambda \in \mathbb{C} \setminus \{0\}$.

On the other hand, based on the parametrization of Q_2 from Proposition 4.4 we know that Q_2 and Q_3 intersect with multiplicity 3 at point $[q : 1 : q^2 + b + cq]$ for some $q \in \mathbb{C}$, $q \neq -\frac{b}{c}$. The line tangent to Q_2 at this point is given by $(2q + c)x + (b - q^2)y - z = 0$, while the line passing through points $[q : 1 : q^2 + b + cq]$ and $[-bc : c^2 : b^2]$ is given by $(b - c^2 - cq)x - b(c + q)y + cz = 0$. Hence by Fact 4.1 c) we obtain yet another equation of Q_3:

$$\begin{aligned} Q_3 : \mu(x^2 + by^2 + cxy - yz) + \big((2q + c)x + (b - q^2)y - z\big)\big((b - c^2 - cq)x - b(c + q)y + cz\big) \\ = (\mu + bc + 2bq - c^3 - 3c^2q - 2cq^2)x^2 + (\mu b - b^2c - b^2q + bcq^2 + bq^3)y^2 - cz^2 \\ + (\mu c + b^2 - 2bc^2 - 4bcq - 3bq^2 + c^2q^2 + cq^3)xy + (2c^2 + 3cq - b)xz \\ + (2bc + bq - cq^2 - \mu)yz = 0 \end{aligned}$$

for some $\mu \in \mathbb{C} \setminus \{0\}$. By comparing the coefficients in both equations for Q_3, we obtain the following system of equations:

$$\begin{cases} \lambda + 2bp - 2cp^2 = \mu + bc + 2bq - c^3 - 3c^2q - 2cq^2, & \text{(I)} \\ bp^3 = \mu b - b^2c - b^2q + bcq^2 + bq^3, & \text{(II)} \\ cp^3 - 3bp^2 = \mu c + b^2 - 2bc^2 - 4bcq - 3bq^2 + c^2q^2 + cq^3, & \text{(III)} \\ 3cp - b = 2c^2 + 3cq - b, & \text{(IV)} \\ bp - cp^2 - \lambda = 2bc + bq - cq^2 - \mu. & \text{(V)} \end{cases}$$

From equation (IV) we obtain $p = \frac{2}{3}c + q$ and we can substitute it into the remaining equations. From equations (I) and (V) we then obtain $\lambda = \mu$ and $q = -\frac{b}{c} - \frac{1}{3}c$ (and hence $p = -\frac{b}{c} + \frac{1}{3}c$). After substituting the obtained values for p and q into equation (III) and simplifying, we get $\mu = -\frac{1}{27}c^3$. Finally, by substituting these values into any of the equation defining Q_3, we obtain the equation from the statement of our proposition. ■

Example 4.13

The arrangement shown below is an affine representation (realized over $\mathbb{R}$) of the arrangement from Proposition 4.12, for $b = 1$, $c = 3$ and with $z = x - y + 1$.

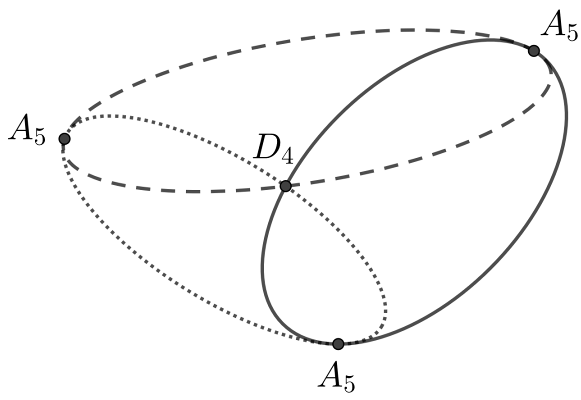


Figure 4.1: The arrangement from Example 4.13

Proposition 4.14

Any free configuration of three smooth conics with weak combinatorics

$$(n_2, t_3, n_3, t_5, d_6, t_7, d_8, d_{10}) = (0, 1, 0, 0, 0, 0, 2, 0)$$

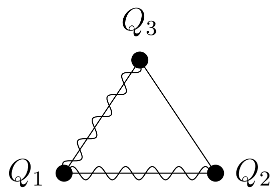
is projectively equivalent to the conics

$$\begin{aligned} &Q_1 : x^2 - yz = 0, \\ &Q_2 : x^2 - \left(\frac{1}{2}c^2 + cp\right) y^2 + cxy - yz = 0, \\ &Q_3 : (5c + 8p)x^2 + 4\left(\frac{1}{2}c + p\right)^3 y^2 - 3(c + 2p)^2 xy - 4xz + (c + 4p)yz = 0 \end{aligned}$$

with $c, p \in \mathbb{C}$ and $c \neq 0$. In this arrangement, the singularities have the following coordinates:

- $[0 : 0 : 1]$ *– a D_8 singularity, where Q_1 and Q_2 are tangent,*
- $\left[\frac{1}{2}c + p : 1 : (\frac{1}{2}c + p)^2\right]$ *– a D_8 singularity, where Q_1 and Q_3 are tangent,*
- $\left[p : 1 : p^2 - \frac{1}{2}c^2\right]$ *– an A_3 singularity, intersection of Q_2 and Q_3.*

Proof. For this arrangement, the only possible decomposition into pairs is $2 \cdot \boldsymbol{\mathcal{A}}^{\mathbf{5}} + \boldsymbol{\mathcal{T}}$. Assume that this configuration of three conics is represented by the following graph:



By Proposition 4.4 we may assume that

$$Q_1 : x^2 - yz = 0, \quad Q_2 : x^2 + by^2 + cxy - yz = 0$$

for some $b, c \in \mathbb{C}$ and $c \neq 0$. These two conics intersect at points $[0:0:1]$ and $[-bc:c^2:b^2]$ with multiplicities 3 and 1, respectively. These two points are the D_8 singularities of our configuration.

The conics Q_1 and Q_3 intersect at points $[0:0:1]$ and $[-bc:c^2:b^2]$ as well, but with respective multiplicities 1 and 3. The line tangent to Q_1 at $[-bc:c^2:b^2]$ is given by the equation $2bcx+b^2y+c^2z=0$, while the line passing through both points is given by $cx+by=0$. Hence by Fact 4.1 c) the equation of Q_3 can be written as

$$\begin{aligned} Q_3 &: \lambda(x^2-yz)+(2bcx+b^2y+c^2z)(cx+by) \\ &= (\lambda+2bc^2)x^2+b^3y^2+3b^2cxy+c^3xz+(bc^2-\lambda)yz=0 \end{aligned}$$

for some $\lambda \in \mathbb{C}\setminus\{0\}$.

The conics Q_2 and Q_3 intersect at points $[0:0:1]$ and $[-bc:c^2:b^2]$ with multiplicity 1. They also intersect at another point with multiplicity 2. From the parametrization of Q_2 from Proposition 4.4 we know that this point has coordinates $[p:1:p^2+b+cp]$ for some $p \in \mathbb{C}$ and $p \neq -\frac{b}{c}$ (otherwise this point coincides with $[-bc:c^2:b^2]$). The line tangent to Q_2 at point $[p:1:p^2+b+cp]$ is given by $(2p+c)x+(b-p^2)y-z=0$. Hence by Fact 4.1 a) we obtain the equation of Q_3 written in the following way:

$$\begin{aligned} Q_3 &: \mu(x^2+by^2+cxy-yz)+\big((2p+c)x+(b-p^2)y-z\big)(cx+by) \\ &= (\mu+2cp+c^2)x^2+(\mu b+b^2-bp^2)y^2+(\mu c+2bc+2bp-cp^2)xy \\ &\quad - cxz-(\mu+b)yz=0 \end{aligned}$$

for some $\mu \in \mathbb{C}\setminus\{0\}$.

We can multiply this equation by $(-c^2)$ so that the coefficient at xz is the same in both equations for Q_3. After that, by comparing the coefficients, we obtain the following system of equations:

$$\begin{cases} \lambda+2bc^2=-c^2(\mu+2cp+c^2), & \text{(I)} \\ b^3=-c^2(\mu b+b^2-bp^2), & \text{(II)} \\ 3b^2c=-c^2(\mu c+2bc+2bp-cp^2), & \text{(III)} \\ bc^2-\lambda=c^2(\mu+b). & \text{(IV)} \end{cases}$$

From equation (IV) we get $\lambda=-\mu c^2$. From equations (I) and (IV) we obtain

$$b=-\frac{1}{2}c^2-cp.$$

Then, by substituting this into equation (II) and factorizing, we get

$$c^3(c+2p)(c^2-4\mu)=0.$$

Hence either $\mu=-\frac{1}{4}c^2$ or $c=-2p$. If $\mu=-\frac{1}{4}c^2$, then $\lambda=\frac{1}{4}c^4$ and equations (I)-(IV) are all satisfied. Substituting these values into equations for Q_2 and Q_3, we obtain the equations from the statement of our proposition. If $c=-2p$, we obtain $b=0$ and from equation (III) we get $\mu=p^2$ (and therefore $\lambda=-4p^4$) – all these values coincide with the previous case with additional assumption $c=-2p$. ■

Example 4.15

The arrangement shown below is an affine representation (realized over $\mathbb{R}$) of the arrangement from Proposition 4.14, for $p=-2$, $c=4$ and with $z=1-x-y$.

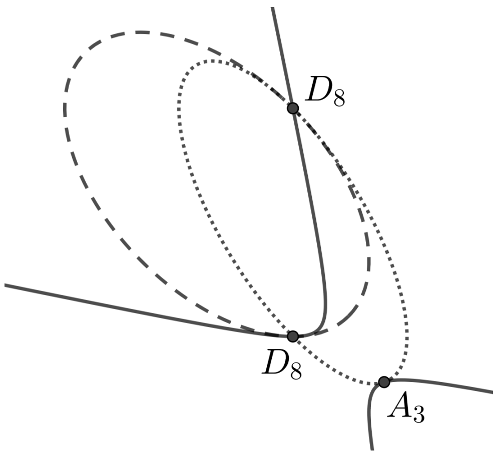


Figure 4.2: The arrangement from Example 4.15

Proposition 4.16

Any free configuration of three smooth conics with weak combinatorics

$$(n_2,t_3,n_3,t_5,d_6,t_7,d_8,d_{10})=(2,1,0,0,0,2,0,0)$$

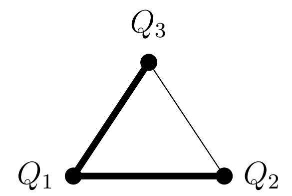
is projectively equivalent to the conics

$$\begin{aligned}
&Q_1: x^2-yz=0,\\
&Q_2: x^2-\frac{1}{2}(p-r)^2z^2-yz=0,\\
&Q_3: (7p^2+2pr-r^2)x^2+2y^2+2p^4z^2-8pxy-8p^3xz+(5p^2-2pr+r^2)yz=0,
\end{aligned}$$

with $p,r\in\mathbb{C}$ and $p\neq r$. In this arrangement, the singularities have the following coordinates:

- $[0:1:0]$ – *an* A_7 *singularity, intersection of* Q_1 *and* Q_2*,*
- $[p:p^2:1]$ – *an* A_7 *singularity, intersection of* Q_1 *and* Q_3*,*
- $\left[p:p^2-\frac{1}{2}(p-r)^2:1\right]$ – *an* A_3 *singularity, intersection of* Q_2 *and* Q_3*,*
- $\left[r:r^2-\frac{1}{2}(p-r)^2:1\right]$ – *an* A_1 *singularity, intersection of* Q_2 *and* Q_3*,*
- $\left[2p-r:\frac{1}{2}(7p^2-6pr+r^2):1\right]$ – *an* A_1 *singularity, intersection of* Q_2 *and* Q_3*.*

PROOF. For this arrangement, the only possible decomposition into pairs is $2\cdot\boldsymbol{\mathcal{A}^7}+\boldsymbol{\mathcal{T}}$. Assume that this configuration of three conics is represented by the following graph:



By Proposition 4.3 we may assume that

$$Q_1 : x^2 - yz = 0, \quad Q_2 : x^2 + az^2 - yz = 0$$

for some $a \in \mathbb{C}\setminus\{0\}$. These conics intersect at point $[0:1:0]$ with multiplicity 4. Based on the parametrization of Q_1 from Proposition 4.3 we know that Q_3 intersects with Q_1 with multiplicity 4 at point $[p:p^2:1]$ for some $p\in\mathbb{C}$. The line tangent to Q_1 at this point is given by the equation $2px - y - p^2z = 0$. Therefore by Fact 4.1 d) we get that the equation of Q_3 can be written in the form

$$\begin{aligned} Q_3 &: \lambda(x^2 - yz) + (2px - y - p^2z)^2 \\ &= (\lambda + 4p^2)x^2 + y^2 + p^4z^2 - 4pxy - 4p^3xz + (2p^2-\lambda)yz = 0 \end{aligned}$$

for some $\lambda\in\mathbb{C}\setminus\{0\}$.

On the other hand, based on the parametrization of Q_2, the conics Q_2 and Q_3 intersect in three points with the coordinates of the form $[q:a+q^2:1]$ (the A_3 singularity), $[r:a+r^2:1]$ and $[w:a+w^2:1]$ (two A_1 singularities) for some pairwise different $q,r,w\in\mathbb{C}$. The line tangent to Q_2 at the A_3 singularity is given by the equation $2qx - y + (a-q^2)z = 0$, while the line passing through both A_1 singularities is given by the equation $(r+w)x - y + (a-rw)z = 0$. Therefore by Fact 4.1 a) we obtain that the equation of Q_3 can be written as

$$\begin{aligned} Q_3 &: \mu(x^2 + az^2 - yz) + \Big(2qx - y + (a-q^2)z\Big)\Big((r+w)x - y + (a-rw)z\Big) \\ &= (\mu + 2qr + 2qw)x^2 + y^2 + (\mu a + a^2 - arw - aq^2 + q^2rw)z^2 \end{aligned}$$

$$
\begin{aligned}
&- (2q + r + w)xy + (ar + aw - q^2r - q^2w + 2aq - 2qrw)xz \\
&+ (q^2 + rw - 2a - \mu)yz = 0
\end{aligned}
$$

for some $\mu \neq 0$. By comparing the coefficients in both equations defining the conic Q_3, we obtain the following system of equations:

$$
\begin{cases}
\lambda + 4p^2 = \mu + 2qr + 2qw, & \text{(I)} \\
p^4 = \mu a + a^2 - arw - aq^2 + q^2rw, & \text{(II)} \\
4p = 2q + r + w, & \text{(III)} \\
-4p^3 = ar + aw - q^2r - q^2w + 2aq - 2qrw, & \text{(IV)} \\
2p^2 - \lambda = q^2 + rw - 2a - \mu. & \text{(V)}
\end{cases}
$$

From equation (III) we get $w = 4p - 2q - r$ and then from equation (I) we obtain $\lambda = \mu - 4p^2 + 8pq - 4q^2$. Subsequently we get $a = -\frac{1}{2}(6p^2 + 3q^2 + r^2 - 8pq - 4pr + 2qr)$ from equation (V). Then, by substituting the obtained values into equation (IV) and factorizing, we obtain

$$2(p - q)(q + r - 2p)^2 = 0,$$

Note that $q + r - 2p \neq 0$, because otherwise equation (III) reduces to $r = w$, which contradicts our assumptions. Hence we have $p = q$ and therefore $w = 2p - r$, $\lambda = \mu$ and $a = -\frac{1}{2}(p - r)^2$. Finally, by substituting all values into equation (II), we obtain

$$\frac{1}{4}(p^2 - 2pr + r^2 + 2\mu)(p - r)^2 = 0,$$

which gives us $\mu = -\frac{1}{2}(p - r)^2$ (because $p \neq r$ would imply $a = 0$). The obtained values can now be substituted into the equations of Q_2 and Q_3 to obtain equations from the statement of the proposition. ■

Example 4.17

The arrangement shown below is an affine representation (realized over $\mathbb{R}$) of the arrangement from Proposition 4.16, for $p = 0$, $r = 1$ and with $z = 1 - y$.

Proposition 4.18

Any free configuration of three smooth conics with weak combinatorics

$$(n_2, t_3, n_3, t_5, d_6, t_7, d_8, d_{10}) = (1, 0, 0, 2, 0, 0, 1, 0)$$

is projectively equivalent to the conics

$$Q_1 : x^2 - yz = 0,$$

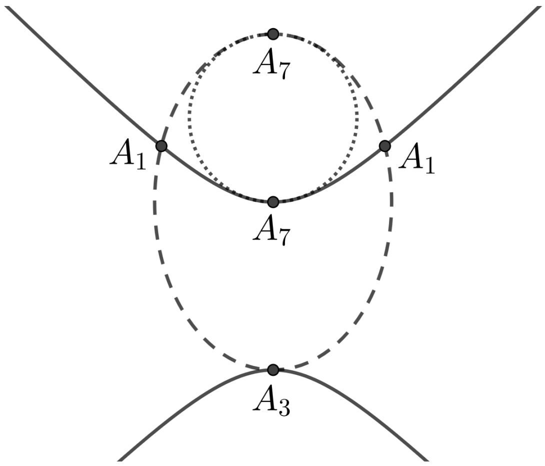


Figure 4.3: The arrangement from Example 4.17

$$Q_2 : x^2 - 3(p-q)(p+\mu)y^2 + 3(p-q)xy - yz = 0,$$
$$Q_3 : (3p - q + \mu)x^2 + p^3y^2 - 3p^2xy - xz + (q-\mu)yz = 0,$$

with $p, q, \mu \in \mathbb{C}$, $p \neq q$ *and* $(p-q)^2 + 3\mu(p-q+\mu) = 0$. *In this arrangement, the singularities have the following coordinates:*

- $[0:0:1]$ – *a* D_8 *singularity, where* Q_1 *and* Q_2 *are tangent,*
- $[p:1:p^2]$ – *an* A_5 *singularity, intersection of* Q_1 *and* Q_3*,*
- $\left[q:1:q^2-3(p-q)(p-q+\mu)\right]$ – *an* A_5 *singularity, intersection of* Q_2 *and* Q_3*,*
- $\left[p+\mu:1:(p+\mu)^2\right]$ – *an* A_1 *singularity, intersection of* Q_1 *and* Q_2*.*

Proof. For this arrangement, the only possible decomposition into pairs is $3 \cdot \boldsymbol{\mathcal{A}}^{\mathbf{5}}$. By Proposition 4.4 we may assume that

$$Q_1 : x^2 - yz = 0, \quad Q_2 : x^2 + by^2 + cxy - yz = 0$$

for some $b, c \in \mathbb{C}$, $c \neq 0$. These two conics intersect at points $[0:0:1]$ and $[-bc : c^2 : b^2]$ with multiplicities 3 and 1, respectively. Assume that the third conic Q_3 passes transversally through the point $[0:0:1]$ (and hence this point becomes the D_8 singularity). Based on the parametrization of Q_1 from Proposition 4.4, the conics Q_1 and Q_3 also intersect with multiplicity 3 at a point with coordinates $[p:1:p^2]$ for some $p \in \mathbb{C}$.

The line tangent to Q_1 at $[p:1:p^2]$ is given by the equation $2px - p^2y - z = 0$ and the line passing through the points $[p:1:p^2]$ and $[0:0:1]$ is given by $x - py = 0$. Therefore by Fact 4.1 c), the equation of Q_3 can be written as

$$\begin{aligned} Q_3 : \lambda(x^2 - yz) &+ (2px - p^2y - z)(x - py) \\ &= (\lambda + 2p)x^2 + p^3y^2 - 3p^2xy - xz + (p-\lambda)yz = 0 \end{aligned}$$

for some $\lambda \in \mathbb{C} \setminus \{0\}$.

On the other hand, based on the parametrization of Q_2 from Proposition 4.4, the conics Q_2 and Q_3 intersect with multiplicity 3 at a point with coordinates $[q : 1 : q^2+b+cq]$ for some $q \in \mathbb{C}$. The line tangent to Q_2 at $[q : 1 : q^2+b+cq]$ is given by $(2q+c)x+(b-q^2)y-z=0$, while the line passing through the points $[0 : 0 : 1]$ and $[q : 1 : q^2+b+cq]$ is given by $x-qy=0$. Therefore by Fact 4.1 c), the equation of Q_3 is of the form

$$\begin{aligned}Q_3 : \mu(x^2+by^2+cxy-yz) &+ \Big((2q+c)x+(b-q^2)y-z\Big)(x-qy)\\ &= (\mu+2q+c)x^2+(\mu b-bq+q^3)y^2+(\mu c+b-3q^2-cq)xy-xz+(q-\mu)yz=0\end{aligned}$$

for some $\mu \in \mathbb{C} \setminus \{0\}$. After comparing the coefficients in both obtained equations for Q_3, we get the following system of equations:

$$\begin{cases} \lambda+2p=\mu+2q+c, & \text{(I)}\\ p^3=\mu b-bq+q^3, & \text{(II)}\\ -3p^2=\mu c+b-3q^2-cq, & \text{(III)}\\ p-\lambda=q-\mu. & \text{(IV)}\end{cases}$$

From equation (IV) we have $\lambda = p-q+\mu$. From equations (I) and (IV) we obtain $c = 3(p-q)$ and then from equation (III) we get $b = -3(p-q)(p+\mu)$. Finally, after substituting the obtained value for b into equation (II) and factorizing the result, we get

$$(p-q)\Big((p-q)^2+3\mu(p-q+\mu)\Big)=0$$

Since $p-q=0$ implies $c=0$, we obtain the condition $(p-q)^2+3\mu(p-q+\mu)=0$ from the statement of our proposition. We also obtain the desired equations by substituting all the obtained values into the equations for conics Q_2 and Q_3. ■

It is easy to verify that the equality $(p-q)^2+3\mu(p-q+\mu)=0$ cannot be satisfied for $p, q, \mu \in \mathbb{R}$ (and $p \neq q$). Therefore this arrangement cannot be realized over $\mathbb{R}$.

Proposition 4.19
Any free configuration of three smooth conics with weak combinatorics

$$(n_2, t_3, n_3, t_5, d_6, t_7, d_8, d_{10}) = (1, 1, 0, 1, 0, 0, 0, 1)$$

is projectively equivalent to the conics

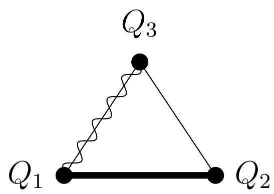
$$\begin{aligned}&Q_1 : x^2-yz=0,\\ &Q_2 : x^2-3(p-q)^2z^2-yz=0,\end{aligned}$$

$$Q_3 : (8p-2q)x^2 + 3p^3z^2 - 3xy - 9p^2xz + (p+2q)yz = 0,$$

with $p, q \in \mathbb{C}$ and $p \neq q$. In this arrangement, the singularities have the following coordinates:

- $[0:1:0]$ – *a D_{10} singularity, where Q_1 and Q_2 are tangent,*
- $[p:p^2:1]$ – *an A_5 singularity, intersection of Q_1 and Q_3,*
- $[q:q^2-3(p-q)^2:1]$ – *an A_3 singularity, intersection of Q_2 and Q_3,*
- $[3p-2q:6p^2-6pq+q^2:1]$ – *an A_1 singularity, intersection of Q_2 and Q_3.*

Proof. For this arrangement, the only possible decomposition into pairs is $\boldsymbol{\mathcal{A}}^7 + \boldsymbol{\mathcal{A}}^5 + \boldsymbol{\mathcal{T}}$. Assume that this configuration of three conics is represented by the following graph:



By Proposition 4.3 we may assume that

$$Q_1 : x^2 - yz = 0, \quad Q_2 : x^2 + az^2 - yz = 0$$

for some $a \in \mathbb{C} \setminus \{0\}$. These two conics intersect at point $[0:1:0]$ (this is a D_{10} singularity in the arrangement of three conics) and Q_3 also passes through this point transversally.

From the parametrization of Q_1 from Proposition 4.3 the conics Q_1 and Q_3 intersect with multiplicity 3 at a point with coordinates $[p:p^2:1]$ for some $p \in \mathbb{C}$. The line tangent to Q_1 at this point is given by the equation $2px - y - p^2z = 0$, while the line passing through points $[0:1:0]$ and $[p:p^2:1]$ is given by $x - pz = 0$. Therefore by Fact 4.1 c) the equation of Q_3 can be written as

$$\begin{aligned} Q_3 &: \lambda(x^2 - yz) + (2px - y - p^2z)(x - pz) \\ &= (\lambda + 2p)x^2 + p^3z^2 - xy - 3p^2xz + (p - \lambda)yz = 0 \end{aligned}$$

for some $\lambda \in \mathbb{C} \setminus \{0\}$.

On the other hand, from the parametrization of Q_2 from Proposition 4.3, the A_3 singularity has coordinates $[q:a+q^2:1]$ for some $q \in \mathbb{C}$, while the remaining points of intersection of Q_2 and Q_3 are $[0:1:0]$ and $[r:a+r^2:1]$ for some $r \in \mathbb{C}$, $r \neq q$. The line tangent to Q_2 at $[q:a+q^2:1]$ is given by $2qx - y + (a-q^2)z = 0$, while the line passing through two other points is given by $x - rz = 0$. Hence by Fact 4.1 a) we obtain of Q_3 in the following form:

$$\begin{aligned} Q_3 &: \mu(x^2 + az^2 - yz) + \Big(2qx - y + (a - q^2)z\Big)(x - rz) \\ &= (\mu + 2q)x^2 + (\mu a - ar + q^2 r)z^2 - xy + (a - q^2 - 2qr)xz + (r - \mu)yz = 0 \end{aligned}$$

for some $\mu \in \mathbb{C} \setminus \{0\}$. By comparing the coefficients in both equations for Q_3, we obtain the following system of equations:

$$\begin{cases} \lambda + 2p = \mu + 2q, & \text{(I)} \\ p^3 = \mu a - ar + q^2 r, & \text{(II)} \\ -3p^2 = a - q^2 - 2qr, & \text{(III)} \\ p - \lambda = r - \mu. & \text{(IV)} \end{cases}$$

From equation (I) we get $\lambda = \mu - 2p + 2q$ and then from (IV) we obtain $r = 3p - 2q$. Hence from equation (III), we get $a = -3(p - q)^2$. After substituting the obtained values into equation (II) and factorizing the result, we obtain

$$(p - q)^2(8p - 8q - 3\mu) = 0.$$

Note that if $p = q$, then $r = 3p - 2q = q$, which contradicts our assumptions. Therefore we obtain $\mu = \frac{8p-8q}{3}$ and as a consequence we get $\lambda = \frac{2p-2q}{3}$. By substituting the obtained values into the equations of Q_2 and Q_3, we obtain the equations from the statement of our proposition. ■

Example 4.20

The arrangement shown below is an affine representation (realized over $\mathbb{R}$) of the arrangement from Proposition 4.19, for $p = 0$, $q = 1$ and with $z = 1 - y$.

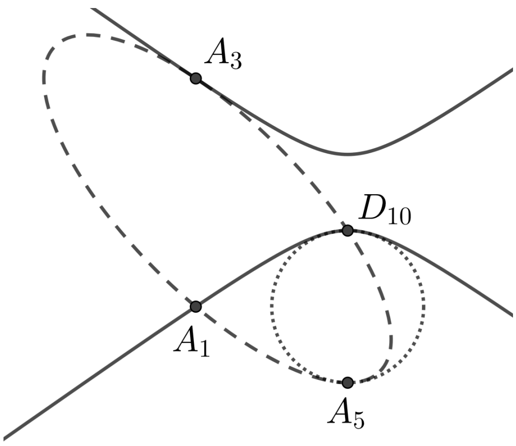


Figure 4.4: The arrangement from Example 4.20

Chapter 5

Appendix

This is the Singular source code, used for computations throughout this thesis. The code is also available at https://github.com/LukaszM111/Singular_code_phd_thesis.

5.1 The second Hessian

The following code is used to calculate the second Hessian of a given curve. It uses the formulas presented in [SS24].

```
option(redSB);
option(noredefine);
proc second_hessian(poly G)
{
    matrix hess[3][3];
    hess[1,1] = diff(diff(G, x), x);
    hess[1,2] = diff(diff(G, x), y);
    hess[1,3] = diff(diff(G, x), z);
    hess[2,1] = diff(diff(G, y), x);
    hess[2,2] = diff(diff(G, y), y);
    hess[2,3] = diff(diff(G, y), z);
    hess[3,1] = diff(diff(G, z), x);
    hess[3,2] = diff(diff(G, z), y);
    hess[3,3] = diff(diff(G, z), z);
    poly H = det(hess);

    list L1;
    L1[1] = hess[2,2]*hess[3,3] - hess[2,3]^2;
    L1[2] = hess[1,1]*hess[3,3] - hess[1,3]^2;
    L1[3] = hess[1,1]*hess[2,2] - hess[1,2]^2;
    L1[4] = hess[1,2]*hess[1,3] - hess[1,1]*hess[2,3];
    L1[5] = hess[1,2]*hess[2,3] - hess[2,2]*hess[1,3];
    L1[6] = hess[1,3]*hess[2,3] - hess[3,3]*hess[1,2];

    list L2;
    L2[1] = diff(diff(H, x), x);
    L2[2] = diff(diff(H, y), y);
    L2[3] = diff(diff(H, z), z);
    L2[4] = 2*diff(diff(H, y), z);
```

```
L2[5] = 2*diff(diff(H, x), z);
L2[6] = 2*diff(diff(H, x), y);

poly OGx = 0;
poly OGy = 0;
poly OGz = 0;
poly OHx = 0;
poly OHy = 0;
poly OHz = 0;
for(int i = 1; i <= 6; i++)
{
    OGx = OGx + diff(L1[i], x) * L2[i];
    OGy = OGy + diff(L1[i], y) * L2[i];
    OGz = OGz + diff(L1[i], z) * L2[i];
    OHx = OHx + L1[i] * diff(L2[i], x);
    OHy = OHy + L1[i] * diff(L2[i], y);
    OHz = OHz + L1[i] * diff(L2[i], z);
}

matrix N[4][4];
N[1,1] = 0;
N[1,2] = diff(H, x);
N[1,3] = diff(H, y);
N[1,4] = diff(H, z);
N[2,1] = diff(H, x);
N[2,2] = hess[1,1];
N[2,3] = hess[1,2];
N[2,4] = hess[1,3];
N[3,1] = diff(H, y);
N[3,2] = hess[2,1];
N[3,3] = hess[2,2];
N[3,4] = hess[2,3];
N[4,1] = diff(H, z);
N[4,2] = hess[3,1];
N[4,3] = hess[3,2];
N[4,4] = hess[3,3];
poly PSI = -det(N);

matrix M[3][3];
M[1,1] = diff(G, x);
M[1,2] = diff(G, y);
M[1,3] = diff(G, z);
M[2,1] = diff(H, x);
M[2,2] = diff(H, y);
M[2,3] = diff(H, z);
M[3,1] = OHx;
M[3,2] = OHy;
M[3,3] = OHz;
poly J1 = det(M);

M[3,1] = OGx;
M[3,2] = OGy;
M[3,3] = OGz;
poly J2 = det(M);
M[3,1] = diff(PSI, x);
M[3,2] = diff(PSI, y);
M[3,3] = diff(PSI, z);
poly J3 = det(M);
```

```
    int d = deg(G);
    poly H2 = (12*d^2 - 54*d + 57)*H*J1 + (d-2)*(12*d - 27)*H*J2 - 20*(d-2)^2*J3;
    return(H2);
}
```

5.2 Minimal polynomial

The following code is used to find a polynomial with $\gamma = f(\alpha, \beta)$ as a root (for any polynomial function f), given that the minimal polynomials of α and β are known. After factorization, it is then possible to determine the minimal polynomial of γ. For instance, it can be used to find the minimal polynomial of $\alpha + \beta$ and $\alpha\beta$.

The algorithm is based on the method presented in the proof of [AS03, Theorem 2.9.1].

```
option(redSB);
option(noredefine);
option(noloadLib);
LIB "linalg.lib";
ring R = 0,(x,u,v),dp;

// Reduce ideal generators
proc calculate(ideal M, poly F, ideal I)
{
    list L;
    for(int i = 1; i <= size(M); i++)
    {
        L = L + list(reduce(M[i]*F, I));
    }
    return(L);
}

// Obtain a polynomial with G as a root from two minimal polynomials F1 and F2
proc get_polynomial ( poly F1, poly F2, poly G )
{
    ideal mono = 1;
    for(int i = 0; i < deg(F1); i++)
    {
        for(int j = 0; j < deg(F2); j++)
        {
            if(i + j > 0)
            {
                mono = mono, u^i*v^j;
            }
        }
    }
    int s = size(mono);
    ideal I = F1, F2;
    I = std(I);
    list L = calculate(mono, G, I);
    matrix A;
    matrix B[s][s];
    for(int i = 1; i <= s; i++)
    {
        A = coef(L[i], uv);
        for(int j = 1; j <= s; j++)
```

```
        {
            for(int k = 1; k <= ncols(A); k++)
            {
                if(mono[j] == A[1,k])
                {
                    B[i,j] = A[2,k];
                }
            }
        }
    }
    poly H = charpoly(B, "x");
    return(H);
}
```

5.3 Division polynomials

The following code is used to compute the division polynomials and then use them to determine the exact coordinates of sextactic points and points of type 9 on F_3.

```
option(noredefine);
ring DP = 0,(x,y,A,B),dp;

// Sequence of division polynomials
poly f(0) = 0;
poly f(1) = 1;
poly f(2) = 2y;
poly f(3) = 3x4 + 6Ax2 + 12Bx - A2;
poly f(4) = 4y * (x6 + 5Ax4 + 20Bx3 - 5A2x2 - 4ABx - 8B2 - A3);
poly f(5) = f(4)*f(2)^3 - f(1)*f(3)^3;
poly f(6) = f(3)*(f(5)*f(2)^2 - f(1)*f(4)^2) /(2y);
poly f(7) = f(5)*f(3)^3 - f(2)*f(4)^3;
poly f(8) = f(4)*(f(6)*f(3)^2 - f(2)*f(5)^2)/(2y);
poly f(9) = f(6)*f(4)^3 - f(3)*f(5)^3;

// Polynomials for Fermat cubic
poly G6 = subst(f(6)/y, A, 0, B, -432);
poly G9 = subst(f(9), A, 0, B, -432);

// Substitution y2 = x3 - 432
proc subst_for_y2(poly F, poly S)
{
    poly G = 0;
    int d = deg(F) div 2;
    for(int i = 0; i <= d; i++)
    {
        G = G + subst(F/(y^(2*i)), y, 0) * S^i;
    }
    return(G);
}
G6 = subst_for_y2(G6, x3-432);
G9 = subst_for_y2(G9, x3-432);

// Computing the coordinates of the sextactic points
// u = 3rd root of unity + cube root of 2
ring R6 = (0,u),x,dp;
minpoly = u6 + 3u5 + 6u4 + 3u3 + 9u + 9;
```

```
poly G = imap(DP, G6);
list X, Y;
list F = factorize(G);
list Fy;
poly f;
for(int i = 2; i <= size(F[1]); i++)
{
    f = -subst(F[1][i], x, 0);
    X = X + list(f, f);
    Fy = factorize(x2 - f^3 + 432);
    Y = Y + list(-subst(Fy[1][2], x, 0), -subst(Fy[1][3], x, 0));
}

// Adding non-trivial 2-torsion points
// These points are of the form (a, 0), where a^3 - 432 = 0
F = factorize(x3 - 432);
for(int i = 2; i <= 4; i++)
{
    X = X + list(-subst(F[1][i], x, 0));
}
Y = Y + list(poly(0), poly(0), poly(0));

// Back to projective coordinates, excluding the 3-torsion points
list C;
poly xp, yp, zp;
for(int i = 1; i <= 35; i++)
{
    xp = (36 + Y[i])/72;
    yp = (36 - Y[i])/72;
    zp = -X[i]/12;
    if(xp*yp*zp != 0)
    {
        C = C + list(xp, yp, zp);
    }
}

// Scale the coordinates so that the x-coordinate is equal to 1
list C1;
for(int i = 1; i <= 27; i++)
{
    C1[3*i-2] = 1;
    C1[3*i-1] = C[3*i-1]/C[3*i-2];
    C1[3*i] = C[3*i]/C[3*i-2];
}

// Coordinates of points of type 6 written with mu and epsilon
ring Rme = 0,(m,e,u),dp;
list C1 = imap(R6, C1);
ideal I = m3 - 2, e2 + e + 1;
I = std(I);
for(int i = 1; i <= 81; i++)
{
    C1[i] = reduce(subst(C1[i], u, m+e), I);
}

// Computing the coordinates of the points of type 9
// u = 9th root of unity + cube root of 3
ring R9 = (0,u),x,dp;
```

```
minpoly = u18 - 15u15 + 177u12 - 578u9 + 6747u6 + 642u3 + 343;
poly G = imap(DP, G9);
list X, Y;
list F = factorize(G);
list Fy;
poly f;
for(int i = 2; i <= size(F[1]); i++)
{
    f = -subst(F[1][i], x, 0);
    X = X + list(f, f);
    Fy = factorize(x2 - f^3 + 432);
    Y = Y + list(-subst(Fy[1][2], x, 0), -subst(Fy[1][3], x, 0));
}

// Back to projective coordinates, excluding the 3-torsion points
list C;
poly xp, yp, zp;
for(int i = 1; i <= 80; i++)
{
    xp = (36 + Y[i])/72;
    yp = (36 - Y[i])/72;
    zp = -X[i]/12;
    if(xp*yp*zp != 0)
    {
        C = C + list(xp, yp, zp);
    }
}

// Scale the coordinates so that the x-coordinate is equal to 1
list C1;
for(int i = 1; i <= 72; i++)
{
    C1[3*i-2] = 1;
    C1[3*i-1] = C[3*i-1]/C[3*i-2];
    C1[3*i] = C[3*i]/C[3*i-2];
}

// Coordinates of points of type 9 written with alpha and beta
ring Rab = 0,(a,b,u),dp;
list C1 = imap(R9, C1);
ideal I = a3 - 3, b6 + b3 + 1;
I = std(I);
for(int i = 1; i <= 216; i++)
{
    C1[i] = reduce(subst(C1[i], u, a+b), I);
}
```

5.4 Bitangent conics

The following code is used to find equations of the bitangent conics, which are tangent to a curve at two points from a given list, with a given multiplicity.

```
option(redSB);
option(noredefine);

// Coefficient at a given monomial
```

```
proc coef_at(poly F, poly vars, poly m)
{
    matrix M = coef(F, vars);
    poly c = 0;
    for(int i = 1; i <= ncols(M); i++)
    {
        if(M[1,i] == m)
        {
            c = M[2,i];
            break;
        }
    }
    return(c);
}

// All monomials of variables m1, m2 of degree k
proc all_monomials(poly m1, poly m2, k)
{
    list L;
    for(int i = 0; i <= k; i++)
    {
        L = L + list(m1^(k-i) * m2^i);
    }
    return(L);
}

// Finding conditions for each point
proc conditions(poly F, int n, poly cx, poly cy, poly cz)
{
    poly K = F;
    poly T = ax2 + by2 + cz2 + dxy + exz + fyz;
    poly m(1..2);
    if(cx != 0)
    {
        cy = cy/cx;
        cz = cz/cx;
        cx = 1;
        K = subst(K, x, 1, y, y + cy, z, z + cz);
        T = subst(T, x, 1, y, y + cy, z, z + cz);
        m(1) = y;
        m(2) = z;
    }
    else
    {
        if(cy != 0)
        {
            cx = cx/cy;
            cz = cz/cy;
            cy = 1;
            K = subst(K, x, x + cx, y, 1, z, z + cz);
            T = subst(T, x, x + cx, y, 1, z, z + cz);
            m(1) = x;
            m(2) = z;
        }
        else
        {
            cx = cx/cz;
            cy = cy/cz;
```

```
            cz = 1;
            K = subst(K, x, x + cx, y, y + cy, z, 1);
            T = subst(T, x, x + cx, y, y + cy, z, 1);
            m(1) = x;
            m(2) = y;
        }
    }
    poly C(1) = subst(T, m(1), 0, m(2), 0);
    T = T - C(1);
    int k = 0;
    poly cf;
    cf = subst(subst(K, m(2), 0)/m(1), m(1), 0);
    if(cf != 0)
    {
        K = K / subst(subst(K, m(2), 0)/m(1), m(1), 0);
        k = 1;
    }
    cf = subst(subst(K, m(1), 0)/m(2), m(2), 0);
    if(cf != 0)
    {
        K = K / subst(subst(K, m(1), 0)/m(2), m(2), 0);
        k = 2;
    }
    if(k == 0)
    {
        return(list());
    }
    int l = 3 - k;
    list L;
    for(int i = 1; i < n; i++)
    {
        L = all_monomials(m(1), m(2), i - 1);
        for(int j = size(L); j >= 1; j--)
        {
            T = T - L[j]*K*subst(T/(L[j]*m(k)), m(l), 0, m(k), 0);
        }
        poly C(i+1) = subst(subst(T, m(k), 0)/(m(l)^i), m(l), 0);
        T = T - C(i+1) * m(l)^i;
    }
    ideal I;
    for(int i = 1; i <= n; i++){ I[i] = C(i); }
    return(I);
}

// Find conics from list of point coordinates
proc find_conics(poly F, list P, int n)
{
    int p = size(P) div 3;
    poly C = ax2 + by2 + cz2 + dxy + exz + fyz;
    list CON;
    list C1;
    list C2;
    ideal I;
    for(int i = 1; i <= p-1; i++)
    {
        for(int j = i+1; j <= p; j++)
        {
            I = conditions(F, n, P[3*i-2], P[3*i-1], P[3*i]),
```

```
            conditions(F, n, P[3*j-2], P[3*j-1], P[3*j]), C;
            I = std(I);
            if(size(I) == 6)
            {
                if(deg(I[6]) > 1)
                {
                    CON = CON + list(I[6]);
                    C1 = C1 + list(i);
                    C2 = C2 + list(j);
                }
            }
        }
    }
    return(list(CON, C1, C2))
}
```

5.5 Point and line arrangements

The following code is a collection of procedures for the study of point and line arrangements. It includes methods for computing the intersection of ideals associated with sets of points, performing projective duality between points and lines, determining the t-vector of a given line arrangement, and applying the Roulleau operators.

```
option(redSB);
option(noredefine);
option(noloadLib);
LIB "elim.lib";

// Make a point ideal from its coordinates
proc make_point(poly xc, poly yc, poly zc)
{
    ideal I = xc*y - yc*x, xc*z - zc*x, yc*z - zc*y;
    return(std(I));
}

// Show point coordinates from a point ideal
proc point_coordinates(ideal P)
{
    ideal I;
    I = P, x - 1;
    I = std(I);
    if(size(I) != 3)
    {
        I = P, y - 1;
        I = std(I);
        if(size(I) != 3)
        {
            I = P, z - 1;
            I = std(I);
        }
    }
    poly xc = -subst(I[3]/(subst(I[3],x,1) - subst(I[3],x,0)), x, 0);
    poly yc = -subst(I[2]/(subst(I[2],y,1) - subst(I[2],y,0)), y, 0);
    poly zc = -subst(I[1]/(subst(I[1],z,1) - subst(I[1],z,0)), z, 0);
    return(list(xc, yc, zc));
```

```
}

// Show coordinates of a list of point ideals
proc point_coordinates_list(list P)
{
    list PC;
    for(int i = 1; i <= size(P); i++)
    {
        PC = PC + list(point_coordinates(P[i]));
    }
    return(PC);
}

// Intersection of all point ideals from the list
proc intersect_all(list L)
{
    int l = size(L);
    ideal I = L[1];
    for(int i = 2; i <= l; i++)
    {
        I = intersect(I, L[i]);
    }
    return(std(I));
}

// Intersection of all point ideals from the list, raised to a power
proc intersect_all_p(list L, int m)
{
    int l = size(L);
    ideal I = L[1]^m;
    for(int i = 2; i <= l; i++)
    {
        I = intersect(I, L[i]^m);
    }
    return(std(I));
}

// As before, but with saturation
proc intersect_all_p_sat(list L, int m)
{
    ideal I = intersect_all(L)^m;
    return(sat(I, ideal(x,y,z))[1]);
}

// Dualize a point to a line
proc point_to_line(ideal P)
{
    list L = point_coordinates(P);
    return(x*L[1] + y*L[2] + z*L[3]);
}

// Dualize a line to a point
proc line_to_point(poly L)
{
    poly xc = subst(L, x, 1, y, 0, z, 0);
    poly yc = subst(L, x, 0, y, 1, z, 0);
    poly zc = subst(L, x, 0, y, 0, z, 1);
    ideal I = x*yc - y*xc, x*zc - z*xc, y*zc - z*yc;
```

```
    return(std(I));
}

// Dualize a list of points
proc dualize_points(list L)
{
    int l = size(L);
    list M;
    for(int i = 1; i <= l; i++)
    {
        M = M + list(point_to_line(L[i]));
    }
    return(M);
}

// Dualize a list of lines
proc dualize_lines(list L)
{
    int l = size(L);
    list M;
    for(int i = 1; i <= l; i++)
    {
        M = M + list(line_to_point(L[i]));
    }
    return(M);
}

// Does a given point lie on a given line?
proc is_on_line(ideal P, poly L)
{
    return(size(reduce(L, std(P))) == 0);
}

// Line through two given points
proc get_line(ideal P1, ideal P2)
{
    ideal I = intersect(P1, P2);
    I = std(I);
    return(I[1]);
}

// Are these points the same?
proc same_points(ideal P1, ideal P2)
{
    return(size(reduce(P1, std(P2))) == 0);
}

// Intersection points and their multiplicities
proc line_arrangement(list L)
{
    list P;
    int new, m;
    int p = 0;
    int l = size(L);
    ideal I;
    for(int i = 1; i <= l-1; i++)
    {
        for(int j = i+1; j <= l; j++)
```

```
        {
            I = L[i], L[j];
            I = std(I);
            if(p > 0)
            {
                new = 1;
                for(int k = 1; k <= p; k++)
                {
                    if(same_points(P[k], I))
                    {
                        new = 0;
                        break;
                    }
                }
                if(new)
                {
                    P = P + list(I);
                    p++;
                }
            }
            else
            {
                P = P + list(I);
                p++;
            }
        }
    }
    list M;
    for(int i = 1; i <= p; i++)
    {
        m = 0;
        for(int j = 1; j <= l; j++)
        {
            if(is_on_line(P[i], L[j]))
            {
                m++;
            }
        }
        M = M + list(m);
    }
    return(list(P, M));
}

// List of lines passing through at least two points
proc lines_from_points(list P)
{
    list L = dualize_points(P);
    list A = line_arrangement(L);
    return(list(dualize_points(A[1]), A[2]));
}

// Pick only points/lines of a given multiplicity
proc pick(int n, list L)
{
    int l = size(L[1]);
    list P;
    for(int i = 1; i <= l; i++)
    {
```

```
        if(L[2][i] == n)
        {
            P = P + list(L[1][i]);
        }
    }
    return(P);
}

// T-vector of a line arrangement
proc t_vector(list L)
{
    list A = line_arrangement(L)[2];
    int m = 2;
    int k;
    int p = 0;
    int l = size(A);
    string s;
    while(p < l)
    {
        k = 0;
        for(int i = 1; i <= l; i++)
        {
            if(A[i] == m){ k++; }
        }
        p = p + k;
        if(k > 0)
        {
            s = "t_" + string(m) + " = " + string(k);
            s;
        }
        m++;
    }
}

// Roulleau P-operator
proc P_operator(list n, list L)
{
    list A = line_arrangement(L);
    int l = size(A[1]);
    int ln = size(n);
    list M;
    int k;
    for(int i = 1; i <= l; i++)
    {
        k = 0;
        for(int j = 1; j <= ln; j++)
        {
            if(A[2][i] == n[j])
            {
                k = 1;
                break;
            }
        }
        if(k)
        {
            M = M + list(A[1][i]);
        }
    }
```

```
    return(M);
}

// Roulleau L-operator
proc L_operator(list n, list P)
{
    list L = dualize_points(P);
    list M = P_operator(n, L);
    return(dualize_points(M));
}
```

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